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Semisimplicity of the DS functor for the orthosymplectic Lie superalgebra

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We prove that the Duflo-Serganova functor DS_x attached to an odd nilpotent element x of $\mathfrak{osp}(m|2n)$ is semisimple, i.e. sends a semisimple representation M of $\mathfrak{osp}(m|2n)$ to a semisimple representation of $\mathfrak{osp}(m-2k|2n-2k)$ where k is the rank of x. We prove a closed formula for $DS_x(L(\lambda))$ in terms of the arc diagram attached to λ.

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1. Introduction

1.1. The Duflo-Serganova functor DS

For a finite dimensional complex Lie superalgebra $\mathfrak{g}$ and an odd element x satisfying $[x,x]=0$ Duflo and Serganova defined a functor $DS_x : Rep(\mathfrak{g}) \to Rep(\mathfrak{g}_x)$ where $\mathfrak{g}_x := ker\ ad(x)/im\ ad(x)$. This functor is given by taking the cohomology of the complex $V \xrightarrow{\rho(x)} V \xrightarrow{\rho(x)} V$ for $(V,\rho) \in Rep(\mathfrak{g})$. For $\mathfrak{gl}(m|n)$ we have $\mathfrak{g}_x \simeq \mathfrak{gl}(m-k|n-k)$ and for $\mathfrak{osp}(m|2n)$ we have $\mathfrak{g}_x \simeq \mathfrak{osp}(m-2k|2n-2k)$ where k is the so-called rank of x. The DS-functor is a symmetric monoidal functor which allows to reduce questions about superdimensions or tensor products to lower rank. It is however very complicated to compute $DS(V)$ explicitly. In [14] the authors derived a closed formula for $DS_x(L(\lambda))$ where $L(\lambda))$ is a finite dimensional irreducible representation of $\mathfrak{gl}(m|n)$ and arbitrary x. In particular $DS_x(L(\lambda))$ is always semisimple.

1.2. Serganova's conjecture

More generally Serganova conjectured [18] that DS_x should be semisimple for any basic classical Lie superalgebra (i.e. $DS_x(L)$ is semisimple for irreducible L). Since the conjecture is trivial in the exceptional cases (in these cases $\mathfrak{g}_x$ is a reductive Lie algebra) this leaves the $\mathfrak{osp}(m|2n)$-case. We prove Serganova's conjecture in this article and give a closed formula for $DS_x(L(\lambda))$ for any x and any λ.

1.3. Main steps and results

We work in the full subcategory $\tilde{\mathcal{F}}(\mathfrak{g})$ of algebraic representations of $\mathfrak{osp}(m|2n)$. The category $\tilde{\mathcal{F}}(\mathfrak{g})$ is canonically isomorphic to the category of $SOSp(m|2n)$-modules; this category decomposes into a direct sum:

$$\tilde{\mathcal{F}}(\mathfrak{g}) = \mathcal{F}(\mathfrak{g}) \oplus \Pi(\mathcal{F}(\mathfrak{g})),$$

where $\mathcal{F}(\mathfrak{g})$ is the full subcategory of modules M, where the parity is induced from the parity on $\Lambda_{m|n}$. We view DS_x as a functor $\tilde{\mathcal{F}}(\mathfrak{g}) \to \tilde{\mathcal{F}}(\mathfrak{g}_x)$. For fixed x of rank r we also write DS_r.

1.3.1. The cases $t = 0, 1, 2$

Recall that $\mathfrak{osp}(M|N)$ consists of two series: the B-series (when M is odd) and the D-series (when M is even). In the $\mathfrak{osp}$-case each block of atypicality k is equivalent to the principal block of $\mathfrak{osp}(2k+t|2k)$ where $t=1$ for the B-series and $t=0,2$ for the D-series. The equivalences are described in [11] (we recall details in 6.2). We consider these three cases separately and set $\mathfrak{g} := \mathfrak{osp}(2m+t|2n)$, where $t=0,1,2$ as above. This division is compatible with DS-functors: if N lies in a block of type t, then $\mathrm{DS}_x(N)$ lies in a block of type t. In Section 6 we reduce the computations of multiplicities $[DS_x(L(\lambda)) : L_{\mathfrak{g}_x}(\nu)]$ to the case of principal blocks.

1.3.2. Reduction to the principal block

Any irreducible module can be moved via a series of translation functors to a *stable* irreducible module (see section 4). The subcategory $\mathcal{F}^g_{st}$ of a block $\mathcal{F}^g(\mathfrak{g})$ consisting of stable modules (a module *stable* if all its simple subquotients are stable) is equivalent via a functor Res to the principal block of $\mathfrak{osp}(2k+t|2k)$ for $t = 0, 1, 2$. Since DS commutes with both Res and translation functors (see section 6.3), we can compute it on irreducible modules in the principal block.

1.3.3. Recursive formulae

We then induct on the degree of atypicality. For atypicality 1 (i.e. the principal block of $\mathfrak{osp}(2|2)$, $\mathfrak{osp}(3|2)$ and $\mathfrak{osp}(4|2)$) $DS(L(\lambda))$ can be computed easily (see for example [9]). In order to treat the general case we establish recursive formulae for the multiplicities of irreducible constituents in $DS_x(L(\lambda))$.

For a category $\mathcal{C}$ denote by $\mathrm{Irr}(\mathcal{C})$ the set of isomorphism classes of simple modules in $\mathcal{C}$. Let V_{st} be the natural representation. For blocks $\tilde{\mathcal{F}}^{g_1}(\mathfrak{g}), \tilde{\mathcal{F}}^{g_1}(\mathfrak{g})$ we denote by $T^{g_0}_{g_1}$ the translation functor

$$T^{g_0}_{g_1} : \tilde{\mathcal{F}}^{g_1}(\mathfrak{g}) \to \tilde{\mathcal{F}}^{g_0}(\mathfrak{g})$$

which maps N to the projection of $N \otimes V_{st}$ to the subcategory $\tilde{\mathcal{F}}^{g_0}(\mathfrak{g})$. Since blocks of atypicality n for $\mathfrak{g}$ and blocks of atypicality $n-r$ for $\mathfrak{g}_x$ $(rk(x) = r)$ correspond to each other (via so-called core diagrams), we may look at

$$T^{g_0}_{g_1} : \tilde{\mathcal{F}}^{g_1}(\mathfrak{g}) \to \tilde{\mathcal{F}}^{g_0}(\mathfrak{g}), \quad T^{g_0}_{g_1} : \tilde{\mathcal{F}}^{g_1}(\mathfrak{g}_x) \to \tilde{\mathcal{F}}^{g_0}(\mathfrak{g}_x).$$

For $N \in \mathcal{F}^{g_1}(\mathfrak{g})$ and $L' \in \mathrm{Irr}(\mathfrak{g}_x)^{g_0}$ we have

$$\begin{aligned}[\mathrm{DS}_r(T^{g_0}_{g_1}(N)) : L'] &= [T^{g_0}_{g_1}(DS_r(N)) : L'] \\ &= \sum_{L_1 \in \mathrm{Irr}(\mathfrak{g}_x)^{g_1}} [DS_r(N) : L_1][T^{g_0}_{g_1}(L_1) : L'].\end{aligned}$$

This formula allows us in Section 7 to successively reduce the computation of the multiplicity $[DS_x(L(\lambda)) : L_{\mathfrak{g}_x}(\nu)]$ to the case where $\mathfrak{g}_x$ is 0 or $\mathbb{C}$.

1.3.4. Multiplicities for the case $\mathrm{rank}\, x = 1$

For such x the condition $\mathfrak{g}_x = 0, \mathbb{C}$ implies $\mathfrak{g} = \mathfrak{osp}(m|2)$ for $m = 2, 3, 4$; for these cases $\mathrm{DS}_x(L)$ can be easily computed. This gives the multiplicity $[\mathrm{DS}_x(L(\lambda)) : L_{\mathfrak{g}_x}(\nu)]$ for $\mathrm{rank}\, x = 1$. As in the $\mathfrak{gl}(m|n)$- and $\mathfrak{p}_n$-cases, treated in [14], [7] respectively, we give the final answer in terms of *arc diagrams*. Notably in all these cases the multiplicities do not depend on x (for x of rank one).

For a weight λ denote by $\mathrm{howl}(\lambda)$ the corresponding weight in the principal block; to $\mathrm{howl}(\lambda)$ we attach an arc diagram $Arc(\mathrm{howl}(\lambda))$, see 8.1.

Theorem A *(see Theorem 8.2 for a more precise statement). Let $rk(x) = 1$. Then $L_{\mathfrak{g}_x}(\nu)$ is a subquotient of $\mathrm{DS}_x(L(\lambda))$ if and only if $Arc(\mathrm{howl}(\nu))$ is obtained from $Arc(\mathrm{howl}(\lambda))$ by removing a maximal arc and, in addition, in the $\mathfrak{osp}(2m+1|2n)$-case, if ν has a sign, then the signs of λ and ν are equal. The multiplicity of $L_{\mathfrak{g}_x}(\nu)$ is either 1 or 2 (depending on the shape of the Arc diagram).*

1.3.5. Semisimplicity for the case $\mathrm{rank}\, x = 1$

In [9] (see also Corollary 6.2.2) it is shown that the extension graph of $\mathcal{F}(\mathfrak{g})$ is bipartite; the bipartition is given by a sign function $\mathrm{dex} : \mathrm{Irr}(\tilde{\mathcal{F}}(\mathfrak{g})) \to \{\pm 1\}$ such that $Ext^1(L_1, L_2) = 0$ if $\mathrm{dex}(L_1) = \mathrm{dex}(L_2)$. Theorem A implies $[\mathrm{DS}_x(L(\lambda)) : L_{\mathfrak{g}_x}(\nu)] = 0$ if $\mathrm{dex}(L(\lambda)) \neq \mathrm{dex}(L(\nu))$ (and $\mathrm{rank}\, x = 1$); this shows the semisimplity of $\mathrm{DS}_x(L(\lambda))$.

1.3.6. σ-Invariance

In the $\mathfrak{osp}(2m|2n)$-case $\mathrm{DS}_x(L(\lambda))$ is invariant with respect to the outer involution σ induced by a Dynkin diagram automorphism of $\mathfrak{g}_x$. This follows from semisimplicity and Theorem A. Alternatively, this can also deduced from the fact that DS (being a tensor functor) commutes with duality (see the proof of Lemma 5.7 (ii)). A similar statement holds for the exceptional Lie superalgebra $F(4)$ with $\mathfrak{g}_x = \mathfrak{sl}_3$, see [9]. A natural question is whether $\mathrm{DS}_x(N)$ is σ-invariant for each finite-dimensional N.

1.4. The general case

We denote with $\mathcal{F}_+(\mathfrak{g})$ the full Serre subcategory of $\tilde{\mathcal{F}}(\mathfrak{g})$ generated by the irreducible objects with $\mathrm{dex}(L) = 1$. By definition it is a semisimple category. We say that a module M is *pure* if for any subquotient L of M, $\Pi(L)$ is not a subquotient of M.

Serganova's semisimplicity conjecture follows from the following theorem.

Theorem B *(Purity and semisimplicity). For each x one has $\mathrm{DS}_x\big(\mathcal{F}_+(\mathfrak{g})\big) = \mathcal{F}_+(\mathfrak{g}_x)$. In particular $DS_x(L(\lambda))$ is pure.*

For each $L \in \mathrm{Irr}(\tilde{\mathcal{F}}(\mathfrak{g}))$ the module $\mathrm{DS}_x(L)$ only depends on the rank of x. Moreover, $\mathrm{DS}_{r+1}(L) \cong \mathrm{DS}_1(\mathrm{DS}_r(L))$, where DS_r stands for DS_x with $\mathrm{rank}\, x = r$.

By above, the statements follow from Theorem A for $\mathrm{rank}\, x = 1$; the general case easily follows by induction on $\mathrm{rank}\, x$.

For $\mathfrak{gl}(m|n)$ purity and semisimplicity were established in [14]; for the exceptional cases purity was checked in [8] (and semisimplicity is trivial since $\mathfrak{g}_x$ is reductive).

By [7], purity holds for the $\mathfrak{p}_n$-case. The other assertions of Theorem B do not hold: DS_1 maps the standard representation of $\mathfrak{p}_2$ to the standard representation of $\mathfrak{p}_1$ which is not semisimple; for a simple $\mathfrak{p}_2$-module L of dimension $(4|4)$, $\mathrm{DS}_2(L) = 0$ whereas

$\mathrm{DS}_1(\mathrm{DS}_1(L)) = \mathbb{C} \oplus \Pi\mathbb{C}$. To the best of our knowledge the composition factors of $\mathrm{DS}_r(L)$ are not known for $r > 1$.

Theorems A, B solve the long-standing problem of computing the superdimension of any irreducible $\mathfrak{osp}(m|2n)$-module (since DS is symmetric monoidal it preserves the superdimensions); see [14] and [7] for analogous results in the $\mathfrak{gl}(m|n)$ and $\mathfrak{p}(n)$-case. The superdimension is up to a sign equal to the dimension of an isotypic representation of $\mathfrak{g}_x$, where $\mathfrak{g}_x$ is either an orthogonal or symplectic Lie algebra or $\mathfrak{osp}(1|2r)$ for some r. Our main theorem also allows to reduce certain questions about tensor products to lower rank similarly to [15], [13]. Our results imply that the DS functor restricts to a symmetric monoidal functor between the full subcategories of indecomposable direct summands in iterated tensor products of irreducible modules. In this world the DS functor can be reinterpreted as a restriction functor similar to [15, Section 5].

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1.6. Index of definitions and notation

Throughout the paper the ground field is $\mathbb{C}$; $\mathbb{N}$ stands for the set of non-negative integers. We denote by Π the parity change functor. In Sections 2—8 $\mathfrak{g} = \mathfrak{osp}(M|2n)$ with $M, n \geq 0$ and M either $2m$ or $2m+1$. One has $\mathfrak{osp}(1|0) = \mathfrak{osp}(0|0) = 0$ and $\mathfrak{osp}(2|0) = \mathbb{C}$.

$\Omega(N), \Sigma, \sigma, \tilde{\mathcal{F}}(\mathfrak{g}), \mathcal{F}(\mathfrak{g}), \mathrm{Irr}, \Lambda^+_{m\|n}, L(\lambda), \chi_\lambda$	Section 2
weight diagram, the diagram $f_- f_+$	3.1
core symbols, core diagram	3.2.1
$\mathcal{F}^g(\mathfrak{g})$	3.3
t, ℓ, $\Lambda^{(t)}_{m+\ell\|n}$, core-free, $\Theta^{(t)}_k$	3.4
$\mathfrak{g}_r, \Sigma_r, S_r$	3.5
tail	3.6
howl, τ	3.7
dex	3.7.8
stable diagrams	3.8
translation functor $T_g^{g'}$	4.1
DS_x	5.2

2. Notation

2.1. Root lattice

Our notation and a choice of triangular decomposition follow [11], [12].

We denote by Δ the set of roots of $\mathfrak{g}$ and by Δ_0 (resp., Δ_1) the set of even (resp., odd) roots. In this paper all modules are weight modules (i.e., $\mathfrak{h}$ acts diagonally) with finite-dimensional weight spaces; for such a module N we set

$$\Omega(N) := \{\nu \in \mathfrak{h}^* | \ N_\nu \neq 0\}.$$

The root system Δ lies in the lattice $\Lambda_{m|n} \subset \mathfrak{h}^*$ spanned by $\{\varepsilon_i\}_{i=1}^m \cup \{\delta_i\}_{i=1}^n$. We denote by Λ the lattice spanned by $\{\varepsilon_i\}_{i=1}^\infty \cup \{\delta_i\}_{i=1}^\infty$ and view $\Lambda_{m|n}$ as a subset of Λ. We define the parity homomorphism $p : \Lambda \to \mathbb{Z}_2$ by $p(\varepsilon_i) = \overline{0}$, $p(\delta_j) = \overline{1}$ for all i, j.

2.2. Triangular decomposition

We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\overline{0}}$. We fix a triangular decomposition corresponding to a "mixed" base Σ, i.e. a base containing maximal possible number of odd roots. For $\mathfrak{osp}(2m+1|2n)$ with $m, n > 0$ we take

$$\Sigma := \begin{cases} \delta_1 - \delta_2, \delta_2 - \delta_3, \ldots, \delta_{n-m} - \varepsilon_1, \varepsilon_1 - \delta_{n-m+1}, \ldots, \varepsilon_m - \delta_n, \delta_n & \text{for } n > m \\ \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{m-n+1} - \delta_1, \delta_1 - \varepsilon_{m-n+2}, \ldots, \varepsilon_m - \delta_n, \delta_n & \text{for } m \geq n \end{cases}$$

and for $\mathfrak{osp}(2m|2n)$ with $m, n > 0$ we take

$$\Sigma := \begin{cases} \delta_1 - \delta_2, \delta_2 - \delta_3, \ldots, \delta_{n-m+1} - \varepsilon_1, \varepsilon_1 - \delta_{n-m+2}, \ldots, \varepsilon_{m-1} - \delta_n, \delta_n \pm \varepsilon_m \\ \quad \text{for } n \geq m \\ \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{m-n} - \delta_1, \delta_1 - \varepsilon_{m-n+1}, \ldots, \varepsilon_{m-1} - \delta_n, \delta_n \pm \varepsilon_m \\ \quad \text{for } m > n \end{cases}$$

For the remaining case $mn = 0$ all triangular decompositions are conjugate and we fix a standard base.

Recall that $\mathfrak{osp}(1|0) = \mathfrak{osp}(0|0) = 0$ and $\mathfrak{osp}(2|0) = \mathbb{C}$; in these cases $\mathfrak{h} = \mathfrak{g}$ and $\Delta = \Sigma = \emptyset$. Note that $\Lambda_{0|0} = 0$ and $\Lambda_{1|0} = \mathbb{Z}\varepsilon_1$.

We denote by ρ the Weyl vector of $\mathfrak{g}$.

2.2.1. Involution σ

The superalgebra $\mathfrak{osp}(2m|2n)$ with $m, n > 0$ or $m > 1, n = 0$ admits an involutive automorphism σ induced by the automorphism of the Dynkin diagram of Σ (the resulting action on $\mathfrak{h}^*$ is given by the reflection r_{ε_m}). Since $\mathfrak{osp}(2m|2n)$ is a Kac-Moody superalgebra, we can choose σ in such a way that

$$\sigma(\mathfrak{g}_{\delta_n \pm \varepsilon_m}) = \mathfrak{g}_{\delta_n \mp \varepsilon_m}, \quad \sigma(\mathfrak{h}) = \mathfrak{h}, \quad \sigma|_{\mathfrak{g}_{\pm\alpha}} = \mathrm{Id} \quad \text{for each } \alpha \in \Sigma \setminus \{\delta_n \pm \varepsilon_m\}. \qquad (1)$$

The restriction of σ to $\mathfrak{o}_{2m} \subset \mathfrak{g}_0$ is the standard involution induced by the automorphism of the Dynkin diagram of $\mathfrak{o}_{2m}$.

For $\mathfrak{osp}(2|0) = \mathbb{C}$ we set $\sigma := -\mathrm{Id}$. For $\mathfrak{osp}(0|2n) = \mathfrak{sp}_{2n}$ we set $\sigma := \mathrm{Id}$. Note that for all cases $\sigma(\Sigma) = \Sigma$.

2.3. Category $\mathcal{F}(\mathfrak{g})$

The category $\mathcal{F}\mathrm{in}(\mathfrak{g})$ of finite dimensional representations of $\mathfrak{g}$ with parity preserving morphisms is the direct sum of two categories: $\tilde{\mathcal{F}}(\mathfrak{g})$ with the modules whose weights lie in $\Lambda_{m|n}$ and $\tilde{\mathcal{F}}^{\perp}(\mathfrak{g})$ with the modules whose weights lie in $\mathfrak{h}^* \setminus \Lambda_{m|n}$. The category $\tilde{\mathcal{F}}^{\perp}(\mathfrak{g})$ is semisimple and $\mathrm{DS}_x(\tilde{\mathcal{F}}^{\perp}(\mathfrak{g})) = 0$ for $x \neq 0$; all simple modules in $\tilde{\mathcal{F}}^{\perp}(\mathfrak{g})$ are typical, their characters are given by the Weyl-Kac character formula.

The category $\tilde{\mathcal{F}}(\mathfrak{g})$ decomposes into a direct sum:

$$\tilde{\mathcal{F}}(\mathfrak{g}) = \mathcal{F}(\mathfrak{g}) \oplus \Pi(\mathcal{F}(\mathfrak{g})),$$

where $\mathcal{F}(\mathfrak{g})$ is the full subcategory with the modules M, where the parity is induced from the parity on $\Lambda_{m|n}$, i.e. $M \in \mathcal{F}(\mathfrak{g})$ if and only if each weight space $p(M_\nu) = p(\nu)$ for all $\nu \in \Lambda_{m|n}$. We will sometimes omit $\mathfrak{g}$ from notation if this does not lead to ambiguity; for instance, we may use $\mathcal{F}$ instead of $\mathcal{F}(\mathfrak{g})$. Except for $\mathfrak{osp}(2|2n)$, $\tilde{\mathcal{F}}(\mathfrak{g})$ is canonically isomorphic to the category of $SOSp(m|2n)$-modules.

2.3.1. For each category $\mathcal{C}$ we denote by $\mathrm{Irr}(\mathcal{C})$ the set of isomorphism classes of irreducible modules in $\mathcal{C}$. For each finite-dimensional $N \in \mathcal{C}$ and $L \in \mathrm{Irr}(\mathcal{C})$ we consider the *graded multiplicity*: we write $[N : L] = (d_0|d_1)$ if a Jordan-Hölder series of N contains d_0 copies of L and d_1 copies of $\Pi(L)$.

We denote by $L(\lambda)$ a simple $\mathfrak{g}$-module of the highest weight λ. We set

$$\Lambda^+_{m|n} := \{\lambda \in \Lambda_{m|n} | \ \dim L(\lambda) < \infty\}.$$

For $\lambda \in \Lambda^+_{m|n}$ we fix the grading in such a way that $L(\lambda) \in \mathcal{F}(\mathfrak{g})$. Then

$$\mathrm{Irr}(\mathcal{F}(\mathfrak{g})) = \{L(\lambda) | \ \lambda \in \Lambda^+_{m|n}\}.$$

We denote by $L_{\mathfrak{g}_x}(\nu)$ the simple $\mathfrak{g}_x$ of the highest weight ν.

We denote by χ_λ the central character of $L(\lambda)$.

3. Weights, roots and diagrams

3.1. Weight diagrams

Many properties of a finite dimensional representation $L(\lambda)$ can be better understood by assigning a *weight diagram* to the weight λ (see e.g. [11]). Note that the conventions how to draw these weight diagrams differ. We follow essentially [11] and list some differences below. These weight diagrams were first introduced in the $\mathfrak{gl}(m|n)$-case in [1] in their work on Khovanov algebras of type A. The weight diagrams introduced in [4] for representations of the orthosymplectic supergroup differ considerably from ours (see [4], Section 6).

3.1.1. Take $\lambda \in \Lambda^+_{m|n}$ and write $\lambda + \rho =: \sum_{i=1}^{m} a_i\varepsilon_i + \sum_{j=1}^{n} b_j\delta_j$.

We assign to λ the (weight) diagram using the following procedure: we label the numberline $\mathbb{N}$ as follows:

for $\mathfrak{osp}(2m|2n)$ we put $>$ (resp., $<$) at the position with the coordinate t if $|a_i| = t$ (resp., $|b_i| = t$) for some i;

for $\mathfrak{osp}(2m+1|2n)$ we put $>$ (resp., $<$) at the position with the coordinate $t - 1/2$ if $|a_i| = t$ (resp., $|b_i| = t$).

If $>, <$ occupy the same position we put the symbol $\times$ ($\times^s$ stands for s symbols $<$ and s symbols $>$; $\underset{\times^s}{>}$ stands for s symbols $<$ and $s+1$ symbols $>$). We put an empty symbol $\circ$ at the non-occupied positions with the coordinates in $\mathbb{N}$. We call the resulting diagram an *unsigned weight diagram.*

For $\mathfrak{osp}(2m|2n)$ with $m \geq 1$ we add the sign $+$ (resp., $-$) if $a_m > 0$ (resp., $a_m < 0$).

For $\mathfrak{osp}(2m+1|2n)$ we put the sign $+$ (resp., $-$) before the diagram if the zero position is occupied by $\times^p$ for $p > 0$ and $(\lambda+\rho|\varepsilon_i) = \frac{1}{2}$ for some i (resp., $(\lambda+\rho|\varepsilon_i) \neq \frac{1}{2}$ for each i).

We call the resulting diagram a *weight diagram* of λ.

Notice that for $\mathfrak{osp}(2m|2n)$-case the action of automorphism $\sigma : \lambda \mapsto \lambda^\sigma$ corresponds to the change of signs of the diagrams; we denote this operation (the change of sign) also by σ.

3.1.2. Note that for $\mathfrak{osp}(2m+1|2n)$-case our weight diagram is obtained from the diagram used in [11] by the shift by $-1/2$.

3.1.3. Examples: $\mathfrak{osp}(2m|2n)$

The empty diagram corresponds to $\mathfrak{osp}(0|0) = 0$; the diagram $>$ (respectively, $-\circ >$) corresponds to the weight 0 (respectively, $-\varepsilon_1$) for $\mathfrak{osp}(2|0) = \mathbb{C}$. One has $L(\emptyset) = \mathbb{C}$ and $L(>)$ is the trivial $\mathfrak{osp}(2|0)$-module.

For $s > 0$ the diagram $\times^s$ is assigned to the weight 0 for $\mathfrak{osp}(2s|2s)$; the diagrams $\overset{\times^s}{>}>, \times^s >>$ are assigned to the $\mathfrak{osp}(2s+4|2s)$-weights 0 and $\varepsilon_1 + \varepsilon_2$ respectively; the diagrams $\pm \circ \times \circ \times$ are assigned to the $\mathfrak{osp}(4|4)$-weights $3(\delta_1 + \varepsilon_1) + (\delta_2 \pm \varepsilon_2)$.

3.1.4. Examples: $\mathfrak{osp}(2m+1|2n)$

The empty diagram corresponds to $\mathfrak{osp}(1|0) = 0$; the diagram $>$ (resp., $<$) corresponds to the weight 0 for $\mathfrak{osp}(3|0) = \mathfrak{o}_3$ (resp., for $\mathfrak{osp}(1|2)$).

For $s > 0$ the diagram $-\times^s$ is assigned to the weight 0 for $\mathfrak{osp}(2s+1|2s)$; the diagram $+\times^s$ is assigned to the weight ε_1 for $\mathfrak{osp}(2s+1|2s)$.

The diagram $\overset{\times^n}{>}>$ is assigned to the $\mathfrak{osp}(2n+5|2n)$-weight 0, the diagrams $\overset{\times}{>}, -\times >$, $+\times >$ to the $\mathfrak{osp}(5|2)$-weights $0, \varepsilon_1, \varepsilon_1+\varepsilon_2$ respectively, and the diagrams $\overset{\times}{<}, -\times <, +\times <$ to the $\mathfrak{osp}(3|4)$-weights $0, \delta_1, \varepsilon_1 + \delta_1$ respectively.

3.1.5. The above procedure gives a one-to-one correspondence between $\Lambda^+_{m|n}$ and the diagrams containing k symbols $\times$, $m-k$ symbols $>$ and $n-k$ symbols $<$ (where $k \leq \min(m,n)$) with the following additional properties:

(i) the coordinates of the occupied position lie in $\mathbb{N}$ and each non-zero occupied position contains exactly one of the signs $\{>, <, \times\}$;
(ii) for $\mathfrak{osp}(2m|2n)$ with $m > 0$ the zero position contains any number of $\times$, no $<$ and at most one $>$; the diagram has a sign if and only if the zero position is empty;
(iii) for $\mathfrak{osp}(2m+1|2n)$ the zero position contains any number of $\times$ and at most one of the symbols $>, <$; the diagram has a sign if and only if the zero position is occupied by $\times^i$ for $i \geq 1$.

3.1.6. The atypicality of λ is equal to the number of symbols $\times$ in the diagram of λ.

3.1.7. Notation

We sometimes identify a dominant weight and its weight diagram; for instance, $f \in \Lambda^+_{m|n}$ means that f is a weight diagram assigned to a weight in $\Lambda^+_{m|n}$.

For a weight diagram f we sometimes use the notation $L(f)$ for the corresponding highest weight module. For instance, $L(\emptyset) = \mathbb{C}$ and $L(>)$ is the trivial $\mathfrak{osp}(2|0)$-module; $L(\times^s)$ (resp., $L(-\times^s)$) stands for the trivial $\mathfrak{osp}(2s|2s)$ (resp., $\mathfrak{osp}(2s+1|2s)$-module) and $L(+\times^s)$ stands for the standard $\mathfrak{osp}(2s+1|2s)$-module.

3.1.8. Remark: OSp-modules

Take $\mathfrak{g} = \mathfrak{osp}(2m|2n)$. By [4, Proposition 4.11] the simple $OSp(2m|2n)$-modules are either of the form $L(\lambda)$ if $\lambda \in \Lambda^+_{m|n}$ is σ-invariant or $L(\lambda) \oplus L(\lambda^\sigma)$. Thus the simple $OSp(2m|2n)$-modules are in one-to-one correspondence with the unsigned $\mathfrak{osp}(2m|2n)$-diagrams. For $\mathfrak{osp}(2m+1|2n)$ and any $\lambda \in \Lambda^+_{m|n}$ there are two irreducible $OSp(2m+1|2n)$-modules $L(\lambda, +)$ and $L(\lambda, -)$ which restrict to $L(\lambda)$.

3.1.9. We denote by f_-f_+ the diagram obtained by "gluing" the diagrams f_- and f_+ (where f_+ does not have sign); for instance,

$$\begin{array}{llll} \overset{\times}{\times} \circ \times = f_-f_+ & \text{where} \ \ f_- = \overset{\times}{\times}\circ, & f_+ = \times \\ + \times^2 \circ\times = f_-f_+ & \text{where} \ \ f_- = +\times^2, & f_+ = \circ\times \end{array}$$

3.2. Core diagrams and central characters

We say that a $\mathfrak{g}$-central character is *dominant* if $\mathcal{F}(\mathfrak{g})$ contains modules with this central character. By [11], the blocks in $\mathcal{F}(\mathfrak{g})$ are parametrized by the dominant central characters and the dominant central characters can be described in terms of typical dominant weights, see below.

3.2.1. We call the symbols $>, <$ the *core symbols*. A *core diagram* is a weight diagram which does not contain symbols $\times$ and does not have $-$ sign.

For a weight diagram f we denote by $\mathrm{core}(f)$ the core diagram which is obtained from the diagram of f by replacing all symbols $\times$ by $\circ$ and by adding the sign $+$ for $\mathfrak{osp}(2m|2n)$-case if the zero position is empty.

For $\lambda \in \Lambda^+_{m|n}$ we denote by $\mathrm{core}(\lambda)$ the weight corresponding to $\mathrm{core}(f)$, where f is the diagram of λ.

For instance, for $\mathfrak{osp}(4|2)$ one has

$$\mathrm{core}(\times >) = \mathrm{core}(\pm\circ > \times) = +\circ >, \quad \mathrm{core}(\varepsilon_1) = \mathrm{core}(\pm\varepsilon_2 + \varepsilon_1 + \delta_1) = \varepsilon_1.$$

Note that for $\mathfrak{osp}(2m|2n)$ one has $\mathrm{core}(\lambda) = \mathrm{core}(\lambda^\sigma)$.

3.2.2. For $\mathfrak{osp}(2m+1|2n)$-case the dominant central characters are parametrized by the core diagrams, i.e. for $\lambda, \nu \in \Lambda^+_{m|n}$

$$\chi_\lambda = \chi_\nu \iff \quad \mathrm{core}(\lambda) = \mathrm{core}(\nu);$$

for $\mathfrak{osp}(2m|2n), \mathfrak{osp}(2m+2|2n)$ one has

$$\chi_\lambda \in \{\chi_\nu, \chi_{\nu^\sigma}\} \iff \quad \mathrm{core}(\lambda) = \mathrm{core}(\nu).$$

(Note that the atypical central characters of $\mathfrak{osp}(2m|2n)$ are σ-invariant: $\chi_\nu = \chi_{\nu^\sigma}$ if ν is atypical.)

3.3. Categories $\mathcal{F}^g(\mathfrak{g})$

For each core diagram g we denote by $\mathcal{F}^g(\mathfrak{g})$ the Serre subcategory of $\mathcal{F}(\mathfrak{g})$[1] generated by $L(\lambda)$ for λ having diagrams f with $\mathrm{core}(f) = g$.

[1] By Serre subcategory generated by a set of simple modules we mean the full subcategory consisting of the modules of finite length whose all simple subquotients lie in a given set.

By above, the categories $\mathcal{F}^g(\mathfrak{osp}(2m+1|2n))$ are the blocks in $\mathcal{F}(\mathfrak{osp}(2m+1|2n))$. For $\mathfrak{g} := \mathfrak{osp}(2m|2n)$ all atypical blocks and σ-invariant typical blocks are of the form $\mathcal{F}^g(\mathfrak{g})$; for a typical block $\mathcal{B}$ with $\mathcal{B} \neq \mathcal{B}^\sigma$ we have $\mathcal{F}^g(\mathfrak{g}) = \mathcal{B} \oplus \mathcal{B}^\sigma$ for a suitable diagram g.

Similarly we use the notation $\tilde{\mathcal{F}}^g(\mathfrak{g})$ for $\mathcal{F}^g(\mathfrak{g}) \oplus \Pi\mathcal{F}^g(\mathfrak{g})$.

3.3.1. Let g be a core diagram with m' symbols $>$ and n' symbols $<$.

The category $\mathcal{F}^g(\mathfrak{osp}(2m+1|2n))$ is non-zero if and only if $m - m' = n - n' \geq 0$ and that $\mathcal{F}^g(\mathfrak{osp}(2m|2n))$ is non-zero if and only if $m - m' = n - n' \geq 0$ and, in addition, g does not have $<$ at the zero position for $m > 0$.

The modules in $\mathcal{F}^g(\mathfrak{g})$ have atypicality $m - m' = n - n'$.

3.4. *Cases* $t = 0, 1, 2$

Recall that $\mathfrak{osp}(M|N)$ consists of two series: B (for odd M) and D. We will distinguish the following cases ($t = 0, 1, 2$):

for the B-series (and any core diagram g) we put $t := 1$;

for the D-series and a core diagram g without $>$ at the zero position we put $t := 0$;

for the D-series and a core diagram g with $>$ at the zero position we put $t := 2$.

3.4.1. We say that a block has type t ($t = 0, 1, 2$) if the core diagram of the corresponding central character has type t. We say that $\lambda \in \Lambda^+_{m|n}$ has type t if the diagram of $\operatorname{core}(\lambda)$ has type t. Then $L(\lambda)$ lies in a block of type t if and only if $\lambda \in \Lambda^+_{m|n}$ has type t.

3.4.2. For $t = 0, 1, 2$ we take $\mathfrak{g} = \mathfrak{osp}(2m + t|2n)$; the weights lattice of $\mathfrak{g}$ is $\Lambda_{m+\ell|n}$, where

$$\ell := \begin{cases} 0 & \text{for } t = 0, 1 \\ 1 & \text{for } t = 2. \end{cases}$$

We denote by $\Lambda^{(t)}_{m+\ell|n}$ the set of dominant weights of type t in $\Lambda^+_{m+\ell|n}$:

$\Lambda^{(1)}_{m+\ell|n} = \Lambda^+_{m|n}$ for the B-series;

$\Lambda^{(0)}_{m+\ell|n}$ is the set of $\mathfrak{osp}(2m|2n)$-dominant weights in $\Lambda^+_{m|n}$ with the diagrams without symbol $>$ at the zero position;

$\Lambda^{(2)}_{m+\ell|n}$ is the set of $\mathfrak{osp}(2m+2|2n)$-dominant weights in $\Lambda^+_{m+1|n}$ with the diagrams having $>$ at the zero position.

We call $\lambda \in \Lambda^{(t)}_{m+\ell|n}$ *core-free* if $\operatorname{core}(\lambda) = \emptyset$ for $t = 0, 1$ and $\operatorname{core}(\lambda) => $ for $t = 2$.

3.4.3. Observe that for $\lambda \in \Lambda^{(t)}_{m+\ell|n}$ we have

$$\lambda + \rho =: \sum_{i=1}^{m+\ell} a_i \varepsilon_i + \sum_{j=1}^{n} b_j \delta_j = \sum_{i=1}^{m} a_i \varepsilon_i + \sum_{j=1}^{n} b_j \delta_j,$$

since for the case $\ell \neq 0$ one has $\ell = 1, a_{m+1} = 0$.

For $t = 0, 1$ the weight $\lambda \in \Lambda^{(t)}_{m+\ell|n}$ has atypicality k if and only if $\mathrm{core}(\lambda) \in \Lambda^+_{m-k|n-k}$. For $t = 2$ the weight $\lambda \in \Lambda^{(t)}_{m+\ell|n}$ has atypicality k if and only if $\mathrm{core}(\lambda) \in \Lambda^+_{m+1-k|n-k}$; note that in this case the core diagram of λ has $>$ at the zero position, so $\mathrm{core}(\lambda)$ lies in $\Lambda^+_{m-k|n-k} \subset \Lambda^+_{m+1-k|n-k}$. Hence in all cases

$$\mathrm{core}(\lambda) \ \text{ is a typical weight in } \Lambda^+_{m-k|n-k}.$$

3.4.4. For $t = 0, 1, 2$ we denote by $\Theta^{(t)}_k$ the set of dominant central characters of atypicality k corresponding to the t-case. By above, the map

$$\chi \mapsto \mathrm{core}(\chi)$$

gives a correspondence between $\Theta^{(t)}_k$ and the set of typical weights $\Lambda^+_{m-k|n-k}$. For $t = 1$ this is a one-to-one correspondence. For $t = 0, 2$ the image consists of the typical weights η satisfying $(\eta|\varepsilon_{m-k}) \neq 0$; the map is injective except for the case $t = 0$ and $k = 0$.

3.5. Algebra $\mathfrak{g}_r$

For $t = 0, 1, 2$ and $r > 0$ we set

$$\mathfrak{g}_r := \mathfrak{osp}(2r + t|2r).$$

Let Σ be the base of simple roots for $\mathfrak{g} = \mathfrak{osp}(2m + t|2n)$. For $0 < r \leq \min(m; n)$ we denote by Σ_r a subset of Σ which is a base of an algebra isomorphic to $\mathfrak{g}_r$ (such Σ_r is unique):

$$\Sigma_r := \begin{cases} \varepsilon_{m-r+1} - \delta_{n-r+1}, \delta_{n-r+1} - \varepsilon_{m-r+2}, \dots, \varepsilon_m - \delta_n, \delta_n & \text{for } t = 1 \\ \delta_{n-r+1} - \varepsilon_{m-r+1}, \dots, \varepsilon_{m-1} - \delta_n, \delta_n - \varepsilon_m, \delta_n + \varepsilon_m & \text{for } t = 0 \\ \varepsilon_{m-r+1} - \delta_{n-r+1}, \delta_{n-r+1} - \varepsilon_{m-r+2}, \dots, \varepsilon_m - \delta_n, \delta_n \pm \varepsilon_{m+1} & \text{for } t = 2. \end{cases}$$

For $r = 0$ we set $\Sigma_r = \emptyset$ and $\rho_r = 0$. Recall that $\mathfrak{osp}(0|0) = \mathfrak{osp}(1|0) = 0$; for $t = 2$ we identify $\mathfrak{osp}(2|0)$ with $\mathbb{C}\varepsilon^*_{m+1} \subset \mathfrak{h}$ (where $\varepsilon^*_m \in \mathfrak{h}$ is such that $\mu(h) = (\mu|\varepsilon_m)$ for each $\mu \in \mathfrak{h}^*$).

We identify $\mathfrak{g}_r$ with the corresponding subalgebra of $\mathfrak{osp}(2m + t|2n)$ (then Σ_r is the base of $\mathfrak{g}_r$). We denote by ρ_r the Weyl vector of $\mathfrak{g}_r$; one has $\rho_r = \rho|_{\mathfrak{g}_r \cap \mathfrak{h}}$ and

$$\rho_r = 0 \ \text{ for } t = 0, 2; \qquad \rho_r = \frac{1}{2} \sum_{i=0}^{r-1} (\delta_{n-i} - \varepsilon_{m-i}) \quad \text{for } t = 1.$$

We denote by S_r the following set: $S_0 = \emptyset$ and

$$S_r := \begin{cases} \{\delta_{n-i} - \varepsilon_{m-i}\}_{i=0}^{r-1} & \text{for } t = 0 \\ \{\varepsilon_{m-i} - \delta_{n-i}\}_{i=0}^{r-1} & \text{for } t = 1, 2. \end{cases}$$

Notice that S_r consists of r isotropic mutually orthogonal roots and $S_r \subset \Sigma_r$.

3.6. Tail

Take $\lambda \in \Lambda^{(t)}_{m+\ell|n}$ and let f be the diagram assigned to λ.

Let $s \leq \min(m, n)$ be the maximal number satisfying $(\lambda|\Sigma_s) = 0$, where Σ_s as above (notice that Σ_s depends on t). We call s the *length of the tail* of λ and write

$$|\operatorname{tail}(\lambda)| = |\operatorname{tail}(f)| := s.$$

For $\mathfrak{osp}(2m|2n)$-case s is equal to the number of symbols $\times$ in the zero position. For $\mathfrak{osp}(2m+1|2n)$ the zero position contains s (resp., $s+1$) symbols $\times$ if the diagram has sign $-$ (resp., $+$); for instance

$$|\operatorname{tail}((-)\times^m)| = |\operatorname{tail}(0)| = m, \qquad |\operatorname{tail}((+)\times^m)| = |\operatorname{tail}(\varepsilon_1)| = m - 1.$$

3.7. Howl

The block of the trivial module is called the *principal block* (there are two principal blocks which differ by Π).

Each block of atypicality k is equivalent to the principal block of $\mathfrak{osp}(2k+t|2k)$. The equivalences are described in [11]; we give some details below. For a dominant weight λ we denote by $\operatorname{howl}(\lambda)$ the corresponding weight in the principal block (roughly speaking, the passage from λ to $\operatorname{howl}(\lambda)$ essentially amounts to removing the core symbols $<, >$ from the weight diagram, see the details below); note that $\operatorname{howl}(\lambda)$ has a core-free diagram.

3.7.1. Let $\lambda \in \Lambda^{(t)}_{m+\ell|n}$ be a weight of atypicality k and let f be the corresponding diagram. For each $i = 1, \ldots, k$ let $s_i(f)$ be the number of the positions to the left of ith symbol $\times$, which do not contain core symbols.

3.7.2. Case $t = 1$

In this case $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$ and the diagram $\operatorname{howl}(f) \in \Lambda^{(1)}_{k|k}$ is the diagram without core symbols, where ith symbol $\times$ occupies $s_i(f)$th position and the sign of $\operatorname{howl}(f)$ is such that the tail lengths of $\operatorname{howl}(f)$ and f are the same. For instance,

$$\begin{gathered} \operatorname{howl}(< \times < \times) = + \times \times; \quad \operatorname{howl}(\pm \times^2 \circ > \times) = \pm \times^2 \circ \times; \\ \operatorname{howl}(\overset{\times}{>} \circ\circ > \times) = - \times \circ \times; \quad \operatorname{howl}(\overset{\times}{>} \times \circ > \times) = + \times^2 \circ \times \end{gathered}$$

and $\operatorname{howl}(f) = \emptyset$ if and only if $k = 0$.

3.7.3. Case $t = 0$

In this case $\mathfrak{g} = \mathfrak{osp}(2m|2n)$ with no $>$ at the zero position. The diagram $\mathrm{howl}(f) \in \Lambda^{(0)}_{k|k}$ is the diagram without core symbols, where the i-th symbol $\times$ occupies the $s_i(f)$-th position and the sign of $\mathrm{howl}(f)$ coincides with the sign of f if $\mathrm{howl}(f)$ requires the sign. For instance,

$$\mathrm{howl}(\pm\circ >> \times) = \pm \circ \times; \quad \mathrm{howl}(\times^2 > \times) = \times^2\times; \quad \mathrm{howl}(\pm\circ > \circ <) = \emptyset.$$

Notice that $|\,\mathrm{tail}(f)| = |\,\mathrm{tail}(\mathrm{howl}(f))|$, so the only case when the diagrams f and $\mathrm{howl}(f)$ do not have the same sign is when $\mathrm{howl}(f) = \emptyset$.

3.7.4. Case $t = 2$

In this case $\mathfrak{g} = \mathfrak{osp}(2m+2|2n)$ and the zero position of f is occupied by $\overset{\times^p}{>}$ $(p \geq 0)$. The diagram $\mathrm{howl}(f) \in \Lambda^{(2)}_{k+1|k}$ has $\overset{\times^p}{>}$ at the zero position; for $i = p+1, \dots, k$ the ith symbol $\times$ in $\mathrm{howl}(f)$ occupies the position $s_i + 1$. For instance,

$$\mathrm{howl}(\overset{\times^2}{>} \circ\circ > \times) = \overset{\times^2}{>} \circ \circ \times$$
$$\mathrm{howl}(> \times < \times) => \times\times; \quad \mathrm{howl}(><) =>$$

3.7.5. For $\lambda \in \Lambda^{(t)}_{m+\ell|n}$ with a diagram f let $\mathrm{howl}(\lambda) \in \Lambda^+$ be the weight corresponding to $\mathrm{howl}(f)$. Notice that for $\lambda \in \Lambda^{(t)}_{m+\ell|n}$ one has $\mathrm{howl}(f) \in \Lambda^{(t)}_{k+\ell|k}$, where k is atypicality of λ.

If $\ell \neq 0$, then $t = 2$ and $\mathrm{howl}(f)$ has $>$ at the zero position, so $\mathrm{howl}(f)$ lies in $\Lambda^+_{k|k}$. Hence in all cases $\mathrm{howl}(f) \in \Lambda^+_{k|k}$.

3.7.6. Note that howl preserves the tail length:

$$|\,\mathrm{tail}(f)| = |\,\mathrm{tail}(\mathrm{howl}(f))|.$$

3.7.7. Connection between the cases $t = 1$ and $t = 2$

Below we describe a remarkable bijection between the core-free diagrams in $\Lambda^{(1)}_{(n|n)}$ and in $\Lambda^{(2)}_{(n|n)}$.

To a diagram $\overset{\times^p}{>} \circ f$ we assign the diagram $-\times^p f$ if $p > 0$ and the diagram $\circ f$ if $p = 0$; to a diagram $\overset{\times^p}{>} \times f$ we assign the diagram $+\times^{p+1} f$.

This assignment gives a bijection τ between the core-free diagrams for $\mathfrak{osp}(2n+2|2n)$ and for $\mathfrak{osp}(2n+1|2n)$. Notice that

$$|\,\mathrm{tail}(\tau(f))| = |\,\mathrm{tail}(f)|.$$

3.7.8. Map dex

We introduce a map $\mathrm{dex} : \Lambda^{(t)}_{m+\ell|n} \to \{\pm 1\}$ by

$$\mathrm{dex}(\lambda) := \begin{cases} (-1)^{p(\mathrm{howl}(\lambda))} & \text{for } t = 0, 1 \\ (-1)^{p(\tau(\mathrm{howl}(\lambda)))} & \text{for } t = 2 \end{cases}$$

(where p is the parity) and the map $\mathrm{Irr}\big(\tilde{\mathcal{F}}(\mathfrak{g})\big) \to \{\pm 1\}$ by

$$\mathrm{dex}(L(\lambda)) := \mathrm{dex}(\lambda), \quad \mathrm{dex}(\Pi(L(\lambda)) = -\,\mathrm{dex}(\lambda).$$

3.8. Stable diagrams

For $\mathfrak{osp}(2m+1|2n)$ a diagram is called *stable* if all symbols $\times$ precede all core symbols $(<, >)$. For $\mathfrak{osp}(2m|2n)$ a diagram is called *stable* if all symbols $\times$ precede all core symbols, except, possibly, the symbol $>$ at the zero position.

A weight λ is called *stable* if the corresponding diagram is stable.

3.8.1. Take $\lambda \in \Lambda^{(t)}_{m+\ell|n}$ and write

$$\lambda + \rho =: \sum_{i=1}^{m} a_i \varepsilon_i + \sum_{j=1}^{n} b_j \delta_j.$$

If λ has atypicality $k > 0$, then

$$\lambda \ \text{ is stable } \iff \mathrm{core}(\lambda) = \sum_{i=1}^{m-k} a_i \varepsilon_i + \sum_{j=1}^{n-k} b_j \delta_j.$$

(The same holds for $k = 0$ with $t = 1, 2$.)

3.8.2. If f is a stable diagram for $\mathfrak{osp}(2m+1|2n)$ (resp., for $\mathfrak{osp}(2m|2n)$) the diagram $\mathrm{howl}(f)$ is obtained from f by replacing all core symbols (resp., all core symbols in the non-zero position) by the empty symbols.

In other words, for λ as above, λ is stable of atypicality k if and only if

$$\mathrm{howl}(\lambda) + \rho_k = \sum_{i=1}^{k} a_{m-k+i} \varepsilon_i + \sum_{j=1}^{k} b_{n-k+j} \delta_j,$$

where ρ_k is the Weyl vector of $\mathfrak{g}_k$.

4. Stabilization

We call a module *stable* if all its simple subquotients are of the form $L(\lambda)$ for stable weights λ. The aim of this section is to show that any module in $\mathcal{F}(\mathfrak{g})$ can be moved with a translation functor to a stable module; these results are known, see [11], [12], but we decided to summarize the proof.

4.1. Translation functors

Let V_{st} be the natural representation. For a core diagrams g, g' we denote by $T_g^{g'}$ the translation functors

$$T_g^{g'}: \mathcal{F}^g(\mathfrak{g}) \to \mathcal{F}^{g'}(\mathfrak{g}), \qquad T_g^{g'}: \tilde{\mathcal{F}}^g(\mathfrak{g}) \to \tilde{\mathcal{F}}^{g'}(\mathfrak{g})$$

which map N to the projection of $N \otimes V_{st}$ to the subcategory $\mathcal{F}^{g'}(\mathfrak{g})$ (resp., $\tilde{\mathcal{F}}^{g'}(\mathfrak{g})$).

We write $T_g^{g'}(f) = f'$ if $T_g^{g'}(L(\lambda)) = L(\lambda')$ and f, f' are the weight diagrams assigned to λ, λ' respectively; similarly, we write $T_g^{g'}(f) = f_1' \oplus f_2'$ if $T_g^{g'}(L(\lambda)) = L(\lambda_1') \oplus L(\lambda_2')$.

4.2. Some useful translation functors

Let $Trans_a$ be the set of the translation functors $T_g^{g'}$, where g is a core diagram with an occupied position a and an empty position $a+1$ and g' is the core diagram obtained from g by interchanging the symbols in the positions $a, a+1$. For example, $Trans_1$ contains $T_{\circ>\circ,}^{\circ\circ>}$, $T_{><\circ<}^{>\circ<<}$ and so on.

Each translation functor $T_g^{g'} \in Trans_a$ is an equivalence of categories except for the case $\mathfrak{osp}(2m|2n)$ with $a = 0$. In the $\mathfrak{osp}(2m+1|2n)$-case (resp., in the $\mathfrak{osp}(2m|2n)$-case) let $Trans$ be the functors which can be written as compositions of functors from $Trans_a$ for all a (resp., $a \neq 0$). All functors in $Trans$ are equivalences of categories.

4.2.1. Case $a \neq 0$

In this case each $T \in Trans_a$ is an equivalence of categories acting on simple modules by interchanging the symbols in positions $a, a+1$ in the corresponding diagrams (and preserving the sign); for example,

$$T_{\circ>\circ}^{\circ\circ>}(\times > *) = \times * >, \quad T_{\circ>\circ,}^{\circ\circ>}(+\circ > *) = + \circ * >$$

where $* \in \{\times, \circ\}$.

4.2.2. Case $\mathfrak{g} = \mathfrak{osp}(2m+1|2n), a = 0$

In this case each $T \in Trans_0$ is an equivalence of categories acting on simple modules by the following rules:

$$> \circ f \mapsto \circ > f \qquad \overset{\times^i}{>} \circ f \mapsto -\times^i > f$$
$$> \times f \mapsto +\times > f \quad \overset{\times^i}{>} \times f \mapsto +\times^{i+1} > f$$

for each $i > 0$ and similar formulae, where $>$ is changed by $<$.

4.3. **Corollary**. *(i) For $T \in Trans$ one has* $\mathrm{howl}(T(f)) = \mathrm{howl}(f)$.

(ii) For any $\lambda \in \Lambda^+_{m|n}, \lambda' \in \Lambda^+_{m-i|n-i}$ satisfying

$$\mathrm{core}(\lambda) = \mathrm{core}(\lambda')$$

there exists $T \in Trans$ such that $T(\lambda), T(\lambda')$ are stable.

(iii) For any module $N \in \mathcal{F}^g(\mathfrak{g})$ there exists $T \in Trans$ such that $T(N)$ is stable.

5. DS-functor

In this section we recall the construction of the DS-functor and describe the algebra $\mathfrak{g}_x$. We prove Corollaries 5.9, 5.10 which will be used later. We distinguish the cases $t = 0, 1, 2$ and take $\mathfrak{g} = \mathfrak{osp}(2m + t|2n)$.

5.1. The DS-functor was introduced in [3]. We recall definitions and some results below. For a $\mathfrak{g}$-module M and $g \in \mathfrak{g}$ we set

$$M^g := \mathrm{Ker}_M\, g.$$

5.2. Construction

We set $\mathfrak{g}_x := \mathfrak{g}^x/[x, \mathfrak{g}]$; note that $\mathfrak{g}^x$ and $\mathfrak{g}_x$ are Lie superalgebras. For a $\mathfrak{g}$-module M we set

$$\mathrm{DS}_x(M) = M^x/xM.$$

Observe that M^x, xM are $\mathfrak{g}^x$-invariant and $[x, \mathfrak{g}]M^x \subset xM$, so $\mathrm{DS}_x(M)$ is a $\mathfrak{g}^{\mathrm{ad}\, x}$-module and $\mathfrak{g}_x$-module. Thus $\mathrm{DS}_x : M \to \mathrm{DS}_x(M)$ is a functor from the category of $\mathfrak{g}$-modules to the category of $\mathfrak{g}_x$-modules.

There are canonical isomorphisms $\mathrm{DS}_x(\Pi(N)) = \Pi(\mathrm{DS}_x(N))$ and

$$\mathrm{DS}_x(M) \otimes \mathrm{DS}_x(N) = \mathrm{DS}_x(M \otimes N).$$

If N is a finite-dimensional $\mathfrak{g}$-module, then $\mathrm{DS}_x(N^*) \cong (\mathrm{DS}_x(N))^*$.

5.2.1. Algebraic representations

The DS functor restricts to a functor

$$DS_x : \tilde{\mathcal{F}}(\mathfrak{g}) \to \tilde{\mathcal{F}}(\mathfrak{g}_x).$$

It does not however preserve the subcategory $\mathcal{F}(\mathfrak{g})$. As already noted in [2] it induces a symmetric monoidal functor between the algebraic representations of $OSp(m|2n)$ and $OSp(m-2r|2n-2r)$.

5.2.2. DS and core diagrams

By [3], Sect. 7 (see also Thm. 2.1 in [17]), the DS-functors preserve the core diagrams, i.e. for a core diagram g one has

$$\mathrm{DS}_x(\mathcal{F}^g(\mathfrak{g})) \subset \tilde{\mathcal{F}}^g(\mathfrak{g}_x), \tag{2}$$

where $\mathfrak{g}_x := \mathrm{DS}_x(\mathfrak{g})$. Warning: $\mathrm{DS}_x(\mathcal{F}^g(\mathfrak{g}))$ is in general not in $\mathcal{F}^g(\mathfrak{g}_x)$, since DS_x does not preserve $\mathcal{F}(\mathfrak{g})$.

5.2.3. DS and translation functors

Since DS is a symmetric monoidal functor

$$\mathrm{DS}_x(N \otimes V_{st}) = \mathrm{DS}_x(N) \otimes \mathrm{DS}_x(V_{st}).$$

Since $\mathrm{DS}_x(V_{st})$ is the natural representation of $\mathfrak{g}_x$, the translation functors "commute with the DS-functors", i.e. the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F}^{g_1}(\mathfrak{g}) & \xrightarrow{T_{g_1}^{g_2}} & \mathcal{F}^{g_2}(\mathfrak{g}) \\ \mathrm{DS}_x \downarrow & & \mathrm{DS}_x \downarrow \\ \tilde{\mathcal{F}}^{g_1}(\mathfrak{g}_x) & \xrightarrow{T_{g_1}^{g_2}} & \tilde{\mathcal{F}}^{g_2}(\mathfrak{g}_x) \end{array}$$

5.3. **Corollary**. *For any $\lambda \in \Lambda^+_{m|n}, \nu \in \Lambda^+_{m-s|n-s}$ with*

$$\mathrm{core}(\lambda) = \mathrm{core}(\nu)$$

there exist stable weights $\lambda_{st} \in \Lambda^+_{m|n}, \nu_{st} \in \Lambda^+_{m-s|n-s}$ such that

$$\mathrm{core}(\lambda_{st}) = \mathrm{core}(\nu_{st});$$
$$\mathrm{howl}(\lambda) = \mathrm{howl}(\lambda_{st}), \qquad \mathrm{howl}(\nu) = \mathrm{howl}(\nu_{st})$$

and $[\mathrm{DS}_x(L(\lambda) : L(\nu)] = [\mathrm{DS}_x(L(\lambda_{st}) : L(\nu_{st})]$ for each x of rank s.

Proof. This follows from Corollary 4.3 and the fact that the translation functors commute with DS_x (see 5.2.2). □

5.4. DS *and automorphisms*

Let $\phi : \mathfrak{g}' \to \mathfrak{g}$ be a homomorphism of Lie superalgebras; for each $\mathfrak{g}$-module N denote by N^ϕ the $\mathfrak{g}'$-module (the vector space N with the action $g'.v := \phi(g')v$).

Each $\phi \in \mathrm{Aut}(\mathfrak{g})$ induces an isomorphism $\overline{\phi} : \mathfrak{g}_x \xrightarrow{\sim} \mathfrak{g}_{\phi(x)}$ and

$$\mathrm{DS}_{\phi^{-1}(x)}(N^\phi) = (\mathrm{DS}_x(N))^{\overline{\phi}}.$$

Let $a \in \mathfrak{g}_0$ be an ad-nilpotent element and $\phi := e^{\mathrm{ad}\, a}$ be the corresponding inner automorphism of $\mathfrak{g}$. If a acts nilpotently on a $\mathfrak{g}$-module N, then $e^a : N \xrightarrow{\sim} N^\phi$. Therefore

$$\mathrm{DS}_{\phi^{-1}(x)}(N) = (\mathrm{DS}_x(N))^{\overline{\phi}}.$$

Let $G_0 = O_{2m+\ell} \times Sp_{2n}$ be the adjoint group of $\mathfrak{g}_{\overline{0}}$, i.e. the subgroup of $\mathrm{Aut}\,\mathfrak{g}$ generated by $e^{\mathrm{ad}\, a}$, where $a \in \mathfrak{g}_0$ is ad-nilpotent. By above, if N is a finite-dimensional $\mathfrak{g}$-module, then

$$\mathrm{DS}_{\phi^{-1}(x)}(N) = (\mathrm{DS}_x(N))^{\overline{\phi}} \tag{3}$$

for any inner automorphism $\phi \in G_0$.

5.5. Choice of x

We call $S \subset \Delta_1$ an *isotropic set* if S is a basis of an isotropic subspace in $\mathfrak{h}^*$. Write $g \in \mathfrak{g}_{\overline{1}}$ as

$$g = \sum_{\alpha \in \mathrm{supp}(g)} g_\alpha,$$

where $g_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$. Set

$$X_{iso} := \{x \in \mathfrak{g}_1 |\ [x, x] = 0\}.$$

We say that $x \in X_{iso}$ has *rank s* if $G_0 x$ contains an element x' such that $\mathrm{supp}(x')$ is an isotropic set of cardinality s. By [3], Sect. 5 for $\mathfrak{osp}(2m|2n)$ with $m > n$ and for $\mathfrak{osp}(2m+1|2n)$ G_0 acts transitively on the set of elements of a fixed rank s; for $\mathfrak{osp}(2m|2n)$ with $m \le n$ this holds for $s < m$ and the elements of rank m form two G_0-orbits which are σ-conjugated.

The rank of $x \in X_{iso}$ is at most $\min(m + \ell, n)$. Assume that the rank of x is greater than m. In this case $\mathfrak{g} = \mathfrak{osp}(2m + 2|2n), t = 2$ and x has rank $m + 1$. However blocks of type $t = 2$ in $\mathcal{F}(\mathfrak{osp}(2m+2|2n))$ have atypicality at most m, so DS_x annihilates the modules in such blocks.

Thus we can (and will) always assume that the rank of x is s, where

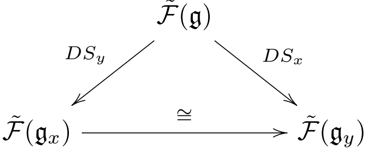
$$0 < s \leq \min(m, n). \tag{4}$$

5.5.1. For each s as in (4) we fix x_s of rank s with

$$\mathrm{supp}(x_s) := S_s$$

and set $\mathrm{DS}_s := \mathrm{DS}_{x_s}$. Notice that $x_s \in \mathfrak{g}_s$.

5.5.2. In general different x (even of the same rank) give rise to different functors DS_x. By (3), x, y induce "the same functor" on $\tilde{\mathcal{F}}(\mathfrak{g})$ if x, y are in the same G_0 orbit, i.e. we have a commutative diagram



with $\mathfrak{g}_x \xrightarrow{\sim} \mathfrak{g}_y$. The description of G_0-orbits in [3] (see 5.5) implies therefore that x, y of the same rank induce "the same functor" on $\tilde{\mathcal{F}}(\mathfrak{g})$ except for $\mathfrak{g} = \mathfrak{osp}(2m|2n)$ with x, y of rank m.

For example, taking a Kac module $K(0)$ of the highest weight zero over $\mathfrak{osp}(2|2) \cong \mathfrak{sl}(2|1)$ we obtain

$$\mathrm{DS}_x(K(0)) \cong \mathrm{DS}_{\sigma(x)}\, K(0)^\sigma = 0, \qquad \mathrm{DS}_{\sigma(x)}(K(0)) \cong \mathrm{DS}_x\, K(0)^\sigma \cong \mathbb{C} \oplus \Pi\mathbb{C}$$

for a suitable x of rank 1.

5.5.3. Remark

By [2] $\mathit{ker}(\mathrm{DS}_1) = \mathit{Proj}$, the thick ideal of projective objects.

5.6. The algebra $\mathfrak{g}_x$

Take $x := x_s$. Set

$$\Delta_x^+ := \{\alpha \in \Delta^+ |\ (\alpha|S_s) = 0\} \setminus S_s$$

and denote by $\mathfrak{g}_x$ the algebra generated by $\mathfrak{g}_{\pm\alpha}$ with $\alpha \in \Delta_x^+$. Clearly, $\mathfrak{g}_x$ is a subalgebra of $\mathfrak{g}^{\mathrm{ad}\, x}$. By [3], $\mathrm{DS}_x(\mathfrak{g}) = \mathfrak{g}^{\mathrm{ad}\, x}/[x, \mathfrak{g}]$ can be identified with $\mathfrak{g}_x$. One has

$$\mathfrak{g}_x \cong \mathfrak{osp}(2(m-s)+t|2(n-s))$$

and $\mathfrak{h}_x := \mathfrak{g}_x \cap \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_x$. The triangular decomposition given by

$$\Delta^+(\mathfrak{g}_x) := \Delta^+_x$$

is of the same form as the triangular decomposition fixed in 2.2. We denote the corresponding base by Σ^x and the Weyl vector by ρ_x.

5.6.1. Examples

Take $s := 2$.

For $\mathfrak{osp}(11|8)$ $(m=5, n=4, t=1)$ we have $\mathfrak{g}_x \cong \mathfrak{osp}(7|4)$ with

$$\begin{aligned}
&\Sigma = \{\varepsilon_1-\varepsilon_2, \varepsilon_2-\delta_1, \delta_1-\varepsilon_3, \varepsilon_3-\delta_2, \delta_2-\varepsilon_4, \varepsilon_4-\delta_3, \delta_3-\varepsilon_5, \varepsilon_5-\delta_4, \delta_4\}\\
&\Sigma^x = \{\varepsilon_1-\varepsilon_2, \varepsilon_2-\delta_1, \delta_1-\varepsilon_3, \varepsilon_3-\delta_2, \delta_2\},\\
&2\rho = \delta_4+\delta_3+\delta_2+\delta_1-\varepsilon_5-\varepsilon_4-\varepsilon_3-\varepsilon_2+\varepsilon_1,\\
&2\rho_x = \delta_2+\delta_1-\varepsilon_3-\varepsilon_2+\varepsilon_1.
\end{aligned}$$

For $\mathfrak{osp}(12|8)$ we have $\mathfrak{g}_x \cong \mathfrak{osp}(8|4)$ and $\rho = \varepsilon_1 = \rho_x$. In this case

$$\Sigma = \{\varepsilon_1-\varepsilon_2, \varepsilon_2-\delta_1, \delta_1-\varepsilon_3, \varepsilon_3-\delta_2, \delta_2-\varepsilon_4, \varepsilon_4-\delta_3, \delta_3-\varepsilon_5, \varepsilon_5-\delta_4, \delta_4\pm\varepsilon_6\};$$

for $t=0$ we have $m=6, n=4$ and

$$\Sigma^x = \{\varepsilon_1-\varepsilon_2, \varepsilon_2-\delta_1, \delta_1-\varepsilon_3, \varepsilon_3-\delta_2, \delta_2\pm\varepsilon_4\};$$

and for $t=2$ we have $m=5, n=4$ and

$$\Sigma^x = \{\varepsilon_1-\varepsilon_2, \varepsilon_2-\delta_1, \delta_1-\varepsilon_3, \varepsilon_3-\delta_2, \delta_2\pm\varepsilon_6\}.$$

5.6.2. Recall that $\mathfrak{h}^*$ has a basis $\{\varepsilon_i\}_{i=1}^{m+\ell} \cup \{\delta_i\}_{i=1}^{n}$ (where $\ell = 0$ for $t = 0, 1$ and $\ell = 1$ for $t = 2$); it is easy to see that $\mathfrak{h}^*_x$ has a basis $\{\varepsilon_i\}_{i=1}^{m-s} \cup \{\delta_i\}_{i=1}^{n-s} \cup \{\varepsilon_{m+\ell}\}$ for $t \neq 0$ and $\{\varepsilon_i\}_{i=1}^{m-s} \cup \{\delta_i\}_{i=1}^{n-s}$ otherwise. For the restriction map $\gamma \mapsto \gamma|_{\mathfrak{h}_x}$ we have

$$\begin{aligned}
&\varepsilon_i \mapsto \varepsilon_i \text{ for } i=1,\dots,m-s; \quad \delta_i \mapsto \delta_i \text{ for } i=1,\dots,n-s;\\
&\varepsilon_{m-i} \mapsto 0, \quad \delta_{n-i} \mapsto 0 \text{ for } i=0,\dots,s-1;
\end{aligned}$$

and, for $\ell = 1$, $\varepsilon_{m+1} \mapsto \varepsilon_{m+1}$. One has $\rho|_{\mathfrak{h}_x} = \rho_x$.

Let $\{\varepsilon'_i\}_{i=1}^{m+\ell-s} \cup \{\delta'_i\}_{i=1}^{n-s}$ be the standard basis in $(\mathfrak{h}')^*$, where $\mathfrak{h}'$ is the Cartan subalgebra of $\mathfrak{osp}(2(m-s)+t|2(n-s))$. The isomorphism $\mathfrak{g}_x \xrightarrow{\sim} \mathfrak{osp}(2(m-s)+t|2(n-s))$ gives $\varepsilon_i \to \varepsilon'_i$ for $i=1,\dots,m-s$, $\delta_j \mapsto \delta'_j$ for $j=1,\dots,n-s$ and $\varepsilon_{m+1} \mapsto \varepsilon_{m+1-s}$ if $\ell = 1$.

For $t = 0, 2$ we denote by σ_x the analogue of σ for $\mathfrak{g}_x$; note that Σ^x is σ_x-invariant. Retain the notation of 2.3.1.

5.7. **Lemma**. *Take* $\mathfrak{g} := \mathfrak{osp}(2m|2n)$.

(i) If N *is finite-dimensional, then* $\mathrm{DS}_x(N^\sigma) \cong (\mathrm{DS}_x(N))^{\sigma_x}$ *if rank* $x < m$;

(ii) Let x *be of rank* 1. *If* L *(resp.,* L'*) is a simple finite-dimensional* $\mathfrak{g}$ *(resp.,* $\mathfrak{g}_x$*)-module, then*

$$[\mathrm{DS}_x(L) : L'] = [\mathrm{DS}_x(L^\sigma) : L'] = [\mathrm{DS}_x(L) : (L')^{\sigma_x}].$$

Remark: The example in 5.5.2 shows that the restriction *rank* $x < m$ is necessary.

Proof. For each s with $0 < s \leq \min(m-1, n)$ take $y \in X_{iso}$ with

$$\mathrm{supp}(y) = \{\varepsilon_{m-i} - \delta_{n+1-i}\}_{i=1}^{s}.$$

Note that y has rank s; by (1), $\sigma(y) = y$. Moreover, σ induces the involution σ_y on the algebra $\mathrm{DS}_y(\mathfrak{g}) \cong \mathfrak{osp}(2(m+\ell-s)|2(n-s))$. Using 5.4 we obtain (i) for $x := y$. By 5.5, this implies (i) for each $x \in G_0 y$ and thus establishes (i) for all x.

Consider the formula in (ii). Notice that $L^* \cong L$ if m is even and $L^* \cong L^\sigma$ if m is odd (since $-\mathrm{Id}$ lies in the Weyl group $W(\mathfrak{osp}(2m|2n))$ if and only if m is even). If m is odd, then $(L')^* \cong L'$ and $\mathrm{DS}_x(L^\sigma) \cong \mathrm{DS}_x(L^*) \cong (\mathrm{DS}_x(L))^*$ which gives the first equality. Similarly, if m is even, we obtain $[\mathrm{DS}_x(L) : L'] = [\mathrm{DS}_x(L) : (L')^{\sigma_x}]$. For $m > 1$, (i) implies the second equality and this establishes the formula. For $m = 1$ one has $\sigma_x = \mathrm{Id}$, so $[\mathrm{DS}_x(L) : L'] = [\mathrm{DS}_x(L) : (L')^{\sigma_x}]$; this completes the proof. □

5.8. **Lemma**. *Let* $\lambda \in \Lambda^{(t)}_{m+\ell|n}$ *be a stable weight of atypicality* k *and let* $x := x_s$. *If* $\nu \in \mathfrak{h}^*$ *satisfies*

$$\nu \leq \lambda; \quad \nu|_{\mathfrak{h}_x} \text{ is dominant}; \quad \mathrm{core}(\lambda) = \mathrm{core}(\nu|_{\mathfrak{h}_x}),$$

then $\nu|_{\mathfrak{h}_x}$ *is stable.*

Proof. Write

$$\lambda + \rho =: \sum_{i=1}^{m+\ell} a_i \varepsilon_i + \sum_{j=1}^{n} b_j \delta_j, \quad \nu + \rho =: \sum_{i=1}^{m+\ell} a'_i \varepsilon_i + \sum_{j=1}^{n} b'_j \delta_j.$$

Set $\nu' := \nu|_{\mathfrak{h}_x}$. Since $\rho|_{\mathfrak{h}_x} = \rho_x$ we obtain

$$\nu' + \rho_x = (\nu + \rho)|_{\mathfrak{h}_x}.$$

Since ν' is dominant and $\mathrm{core}(\lambda) = \mathrm{core}(\nu')$ we have $\nu' \in \Lambda^{(t)}_{m+\ell-s|n-s}$. If $\ell = 1$ we have $t = 2$ and thus $a'_{m+1} = 0$. Therefore for all t we have

$$\nu' + \rho_x = (\nu + \rho)|_{\mathfrak{h}_x} = \sum_{i=1}^{m-s} a'_i \varepsilon_i + \sum_{j=1}^{n-s} b'_j \delta_j.$$

If $k = 0$, then $s = 0$ and $\nu' = \nu$ is stable. If ν' is typical, it is stable. Thus we assume that λ, ν' are atypical. Since λ is stable, 3.8.1 gives

$$\text{core}(\nu') = \text{core}(\lambda) = \sum_{i=1}^{m-k} a_i \varepsilon_i + \sum_{j=1}^{n-k} b_j \delta_j. \tag{5}$$

Moreover, by 3.8.1, for stability of ν' it is enough to verify that $a_i = a'_i$ for $i = 1, \ldots, m-k$ and $b_j = b'_j$ for $j = 1, \ldots, n-k$. Let p (resp., q) be minimal such that $a_p \neq a'_p$ (resp., $b_q \neq b'_q$). By above it suffices to show that

$$m - p, n - q < k. \tag{6}$$

Using (5) (and the atypicality of ν' for $t = 0$ case) we obtain

$$a_p \in \{a'_i\}_{i=p}^{m-s} \quad \text{if } p \leq m-k; \qquad b_q \in \{b'_j\}_{i=q}^{n-s} \quad \text{if } q \leq n-k. \tag{7}$$

The assumption $\nu \leq \lambda$ gives

$$\lambda - \nu = (a_p - a'_p)\varepsilon_p + (b_q - b'_q)\delta_q + \sum_{i>p}(a_i - a'_i)\varepsilon_i + \sum_{j>q}(b_j - b'_j)\delta_j \in \mathbb{N}\Sigma. \tag{8}$$

Consider the case when $\varepsilon_p - \delta_q \in \Delta^+$. Then (8) implies $a_p > a'_p$. Since ν' is dominant

$$a'_i \leq a'_p < a_p \quad \text{for } i = p, p+1, \ldots, m-s,$$

so $p \geq m - k$ by (7). Notice that $\delta_{n-j} - \varepsilon_{m-i} \in \Delta^+$ for $j > i$, so the assumption $\varepsilon_p - \delta_q \in \Delta^+$ gives $n-q \leq m-p$ and thus implies (6). For the remaining case $\delta_q - \varepsilon_p \in \Delta^+$ the proof is similar. □

Taking $s = 0$ we obtain the following corollary, which is a reformulation of Lemma 6.2 in [17].

5.9. **Corollary**. *Assume that λ, ν are dominant weights, λ is stable and*

$$\text{core}(\lambda) = \text{core}(\nu) \qquad \nu \leq \lambda.$$

Then ν is stable.

5.10. **Corollary**. *If $N \in \mathcal{F}^g$ is stable, then* $\mathrm{DS}_x(N)$ *is stable.*

Proof. Assume that $[\mathrm{DS}_x(N) : L_{\mathfrak{g}_x}(\nu')] \neq 0$. Then ν' is dominant and

$$\mathrm{core}(\nu') = g.$$

Let $\overline{v}$ be a vector in $\mathrm{DS}_s(N) = N^x/xN$ which has weight ν'. By [8], Lem. 2.3 we can choose a preimage v of $\overline{v}$ in the space

$$\sum_{\mu \in X} N^x_\mu, \quad \text{where } X := \{\nu \in \Omega(N) |\ \nu|_{\mathfrak{h}_x} = \nu', \ \ (\nu|S_s) = 0\}.$$

Take $\nu \in X$. Since $\nu \in \Omega(N)$ there exists a stable dominant weight λ (a maximal weight in $\Omega(N)$) such that

$$\nu \leq \lambda, \qquad \mathrm{core}(\lambda) = g.$$

In the light of 5.6.2 the condition $(\nu|S_s) = 0$ implies

$$\mathrm{core}(\nu) = \mathrm{core}(\nu|_{\mathfrak{h}_x}) = g.$$

By Lemma 5.8, ν' is stable. □

6. Reduction to principal blocks

In this section we reduce the computation of multiplicities $[\mathrm{DS}_s(L(\lambda)) : L_{\mathfrak{g}_x}(\nu')]$ to the case of principal blocks.

In this section g stands for a core diagram of type t for $\mathfrak{g} := \mathfrak{osp}(2m+t|2n)$. Let $\mu \in \Lambda^{(t)}_{m-k|n-k}$ be the typical weight corresponding to g.

6.1. Notation

We denote by $\mathcal{F}^{pr}(\mathfrak{g}_k)$ the principal block for $\mathfrak{g}_k$:

$$\mathcal{F}^{pr}(\mathfrak{g}_k) = \begin{cases} \mathcal{F}^{\emptyset}(\mathfrak{osp}(2k+t|2k) & \text{for } t = 0, 1 \\ \mathcal{F}^{>}(\mathfrak{osp}(2k+2|2k) & \text{for } t = 2. \end{cases}$$

We denote by $\mathcal{F}^g_{st}$ the subcategory of $\mathcal{F}^g(\mathfrak{g})$ consisting of stable modules. For $t = 0, 1$ this category is zero if and only if the zero position of g is non-empty; for $t = 2$ this category is always non-zero.

We denote by $\Lambda^+(g; i)$ the set of diagrams f with the following properties:

core$(f) = g$ and all symbols $\times$ lie in the positions $0, \dots, i$.

We denote by $\mathcal{F}^g_i(\mathfrak{g})$ the Serre subcategory of $\mathcal{F}(\mathfrak{g})$ generated by the modules $L(\lambda)$ with $\lambda \in \Lambda^+(g; i) \cap \Lambda_{m|n}$ and denote by $\mathcal{F}^{pr}_i(\mathfrak{g}_k)$ the corresponding subcategory of $\mathcal{F}^{pr}(\mathfrak{g}_k)$. Note that $\mathcal{F}^g(\mathfrak{g})$ can be viewed as a "limit" of subcategories $\mathcal{F}^g_i(\mathfrak{g})$.

6.2. The functor Res *for* $(t;k) \neq (0;0)$

We assume that $\mathcal{F}^g(\mathfrak{g})$ is a non-principal block and $\mathcal{F}^g_{st}(\mathfrak{g}) \neq 0$. This means that we exclude the case $t = 0, k = 0, m > 0$, since in this case $\mathcal{F}^g(\mathfrak{g})$ is a direct sum of two blocks, and that g has a non-empty symbols at a non-zero position. One has

$$\mathcal{F}^g_{st}(\mathfrak{g}) = \mathcal{F}^g_q(\mathfrak{g}),$$

where $q+1$ is the coordinate of the first occupied non-zero position in g.

We retain the notation of A.1. Fix $z \in \mathfrak{h}$ such that $\alpha(z) = 0$ for $\alpha \in \Delta(\mathfrak{g}_k)$ and $\alpha(z) \in \mathbb{R}_{>0}$ for $\alpha \in \Delta^+ \setminus \Delta(\mathfrak{g}_k)$. Then

$$\mathfrak{g}^z = \mathfrak{g}_k + \mathfrak{h} = \mathfrak{g}_k \times \mathfrak{h}'',$$

where $\mathfrak{h}''$ is the centralizer of $\mathfrak{g}_k$. Set $a := (\mu - \rho)(z)$ and define the functor $\mathrm{Res} := \mathrm{Res}_a$ using the construction of A.2 for $\mathfrak{l} := \mathfrak{g}_k$ and μ as above.

6.2.1. **Proposition**. *The functor*

$$\mathrm{Res}: \ \mathcal{F}^g_{st}(\mathfrak{g}) \xrightarrow{\sim} \mathcal{F}^{pr}_q(\mathfrak{g}_k)$$

is an equivalence of the categories and

$$\mathrm{Res}(L(f)) = L(\mathrm{howl}(f))$$

for each stable $f \in \Lambda^{(t)}_{m+\ell|n}$ *with* $\mathrm{core}(f) = g$.

Proof. Take $\mathfrak{h}' := \mathfrak{g}_k \cap \mathfrak{h}$. Then $\mathfrak{h} = \mathfrak{h}' \times \mathfrak{h}''$. Setting

$$E := \{\varepsilon_i\}_{i=1}^{m+\ell}, \quad D := \{\delta_i\}_{i=1}^{n}, \quad E' := \{\varepsilon_i\}_{i=m+1-k}^{m+\ell}, \quad D' := \{\delta_i\}_{n+1-k}^{n},$$

we see that $(\mathfrak{h}')^*$ is spanned by $E' \cup D'$ and $(\mathfrak{h}'')^*$ is spanned by $(E \setminus E') \cup (D \setminus D')$.

Let $\upsilon : \mathfrak{h}^* \to (\mathfrak{h}')^*$, $\upsilon'' : \mathfrak{h}^* \to (\mathfrak{h}'')^*$ be the projections given by the decomposition $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$. Notice that $\mu \in (\mathfrak{h}'')^*$ and set

$$A := \{\lambda \in \Lambda^+_{m+\ell|n} | \ \mathrm{core}(\lambda) = g, \ \upsilon''(\lambda) = \mu - \upsilon''(\rho)\}, \quad A' := \upsilon(A).$$

One has

$$\begin{aligned} A &= \{\lambda \in \Lambda^+_{m+\ell|n} | \ \mathrm{core}(\lambda) = g, \ \upsilon''(\lambda) = \mu - \upsilon''(\rho)\} \\ &= \{\lambda \in \Lambda^+_{m+\ell|n} | \ \mathrm{core}(\lambda) = g, \ \upsilon''(\lambda + \rho) = \mu\}. \end{aligned}$$

In the light of 3.8.1 we get $A = \Lambda^+_{m+\ell|n}(g;q)$, so $\mathcal{F}^g_{st}(\mathfrak{g}) = \mathcal{F}^g_q(\mathfrak{g}) = \mathcal{F}(A)$. By Corollary 5.9, $\Lambda^+_{m+\ell|n}(g;q)$ satisfies the assumption (22). By A.5, Res provides an equivalence of categories $\mathcal{F}^g_{st}(\mathfrak{g}) \xrightarrow{\sim} \mathcal{F}(A')$.

Take $\lambda \in A$. Since λ is stable, 3.8.2 implies that

$$\lambda + \rho = \mu + \text{howl}(\lambda) + \rho_k,$$

where ρ_k is the Weyl vector of $\mathfrak{g}_k$. Note that $\rho_k = \upsilon(\rho)$.

Since $\mu \in (\mathfrak{h}'')^*$ and $\text{howl}(\lambda) \in (\mathfrak{h}')^*$, we have $\upsilon(\lambda) = \text{howl}(\lambda)$, so

$$A' = \upsilon(\Lambda^+_{m+\ell|n}(g; q)) = \Lambda^+(g_{pr}; q),$$

where g_{pr} is empty for $t = 0, 1$ and $g_{pr} =>$ for $t = 2$. Hence $\mathcal{F}(A') = \mathcal{F}^{pr}_q(\mathfrak{g}_k)$. This completes the proof. $\square$

6.2.2. Retain the notation of 3.7.8.

Corollary. *For each* $\lambda, \nu \in \Lambda^{(t)}_{m|n}$ *one has*

$$\text{dex}(\lambda) = \text{dex}(\nu) \quad \Longrightarrow \quad \text{Ext}^1(L(\lambda), L(\nu)) = 0.$$

Proof. For the core-free diagrams the assertion is established in [9]. The general case follows from Proposition 6.2.1 (note that $\text{Ext}^1(L(\lambda), L(\nu)) = 0$ if λ is typical). $\square$

6.2.3. Remark

Consider the case $k = 0$ (and $t \neq 0$). In this case $\mathcal{F}^g_{st}(\mathfrak{g}) = \mathcal{F}^g(\mathfrak{g})$ is a typical block containing $L(\mu)$; this block is isomorphic to the category of finite-dimensional even vector spaces.

For $t = 1$ one has $\mathfrak{g}_k = 0$ and $\mathcal{F}^{pr}_q(\mathfrak{g}_k) = \mathcal{F}(\mathfrak{g}_k)$ is the category of finite-dimensional even vector spaces ($\text{Res}(L(\mu)) = L(\emptyset) = \mathbb{C}$).

For $t = 2$ one has $\mathfrak{g}_k = \mathbb{C}$ and $\mathcal{F}(\mathfrak{g}_k)$ is the category of finite-dimensional $\mathfrak{g}_k$-module with the zero action of $\mathfrak{g}_k$ (since $t = 2$), i.e. ($\text{Res}(L(\mu))$ is the trivial $\mathfrak{g}_k$-module).

6.3. DS_x *and* Res

Take $x := x_s$. Since $\text{DS}_s(\mathcal{F}^g) = 0$ for $s > k$ we assume

$$0 < s \leq k.$$

By 5.5.1 one has $x \in \mathfrak{g}_s \subset \mathfrak{g}_k$.

Let $(\mathfrak{g}_x)_i$ be the subalgebra of $\mathfrak{g}_x$ constructed for $\mathfrak{g}_x$ in the same way as $\mathfrak{g}_i$ to $\mathfrak{g}$. Consider the base $\Sigma^x_{k-s} \subset \Delta^+(\mathfrak{g}_x)$. We define the functors Res, Res_x as in 6.2: we take

$$\begin{array}{llll} z \in \mathfrak{h} \ \text{ such that } & \alpha(z) = 0 \ \text{ for } \alpha \in \Sigma_k, & \alpha(z) = 1 \ \text{ for } \alpha \in \Sigma \setminus \Sigma_k \\ z_x \in \mathfrak{h}_x \ \text{ such that } & \alpha(z_x) = 0 \ \text{ for } \alpha \in \Sigma^x_{k-s}, & \alpha(z_x) = 1 \ \text{ for } \alpha \in (\Sigma^x \setminus \Sigma^x_{k-s}) \end{array}$$

and set $a := (\mu - \rho)(z)$, $a_x := (\mu - \rho_x)(z_x)$. We obtain $\text{Res} : \mathcal{F}^g_{st}(\mathfrak{g}) \longrightarrow \mathcal{F}^{pr}_q(\mathfrak{g}_k)$ and

$$\mathrm{Res}_x : \tilde{\mathcal{F}}^g_{st}(\mathfrak{g}_x) \longrightarrow \tilde{\mathcal{F}}^{pr}_q((\mathfrak{g}_x)_{k-s})$$

given by

$$\mathrm{Res}(N) := \{v \in N|\ zv = a\}, \qquad \mathrm{Res}_x(N) := \{v \in N|\ z_x v = a_x v\}.$$

6.3.1. **Lemma**. *One has* $\mathrm{DS}_x(\mathfrak{g}_k) = (\mathfrak{g}_x)_{k-s}$. *In addition,* $z_x = z$, $a_x = a$ *except for the case* $t = 0, k = s$.

Proof. The partial order $\geq$ gives a total order on the standard basis of $\mathfrak{h}^*$, i.e.

$$\{\varepsilon_i\}_{i=1}^{m+\ell} \cup \{\delta_j\}_{j=1}^{n} = \{e_i\}_{i=1}^{m+n+\ell}, \quad e_1 < e_2 < \ldots < e_{m+n+\ell}$$

(where $e_1 = \delta_n$ for $\mathfrak{osp}(2m+1|2n)$ and $e_1 = \varepsilon_{m+\ell}$ for $\mathfrak{osp}(2m+2|2n)$). Let $\{e_i^*\}_{i=1}^{m+n+\ell}$ be the dual basis of $\mathfrak{h}$.

Denote the base of $\mathrm{DS}_x(\mathfrak{g}_k)$ by $(\Sigma_k)^x$. For $k = s$ we have $(\Sigma_k)^x = \emptyset = (\Sigma^x)_0$.

In the $t = 0$-case one has

$$\begin{array}{ll}
\Sigma = \{e_1 + e_2, e_2 - e_1, e_3 - e_2, \ldots, e_{m+n} - e_{m+n-1}\}, & \\
\Sigma_k = \{e_1 + e_2, e_2 - e_1, \ldots, e_{2k} - e_{2k-1}\} & \\
\Sigma^x = \{e_{2s+1} + e_{2s+2}, e_{2s+2} - e_{2s+1}, \ldots, e_{m+n} - e_{m+n-1}\} & \text{if } s < m \\
\Sigma^x = \{2e_{2s+1}, e_{2s+2} - e_{2s+1}, \ldots, e_{m+n} - e_{m+n-1}\} & \text{if } s = k = m \\
(\Sigma^x)_{k-s} = \{e_{2s+1} + e_{2s+2}, e_{2s+2} - e_{2s+1}, \ldots, e_{2k} - e_{2k-1}\} = (\Sigma_k)^x. &
\end{array}$$

In the $t = 1$-case one has

$$\begin{array}{l}
\Sigma = \{e_1, e_2 - e_1, e_3 - e - 2, \ldots, e_{m+n} - e_{m+n-1}\}, \\
\Sigma_k = \{e_1, e_2 - e_1, \ldots, e_{2k} - e_{2k-1}\} \\
\Sigma^x = \{e_{2s+1}, e_{2s+2} - e_{2s+1}, e_{2s+3} - e_{2s+2}, \ldots, e_{m+n} - e_{m+n-1}\} \\
(\Sigma^x)_{k-s} = \{e_{2s+1}, e_{2s+2} - e_{2s+1}, \ldots, e_{2k} - e_{2k-1}\} = (\Sigma_k)^x.
\end{array}$$

In both cases

$$\Sigma \setminus \Sigma_k = \{e_{2k+1} - e_{2k}, e_{2k+2} - e_{2k+1}, \ldots, e_{m+n} - e_{m+n-1}\}$$

and

$$z = e^*_{2k+1} + 2e^*_{2k+2} + \ldots + (m+n-2k)e^*_{m+n}.$$

One readily sees that $z \in \mathfrak{h}_x$ and that $z = z_x$ except for the case $t = 0$, $k = s$.

In the $t = 2$-case one has

$$\begin{array}{l}
\Sigma = \{e_1 + e_2, e_2 - e_1, e_3 - e_2, \ldots, e_{m+n+1} - e_{m+n}\}, \\
\Sigma_k = \{e_1 + e_2, e_2 - e_1, \ldots, e_{2k} - e_{2k-1}, e_{2k+1} - e_{2k}\} \\
\Sigma^x = \{e_1 + e_{2s+2}, e_{2s+2} - e_1, e_{2s+3} - e_{2s+2}, \ldots, e_{m+n+1} - e_{m+n}\} \\
(\Sigma^x)_{k-s} = \{e_1 + e_{2s+2}, e_{2s+2} - e_{2s+1}, \ldots, e_{2k+1} - e_{2k}\} = (\Sigma_k)^x.
\end{array}$$

Therefore

$$z = e^*_{2k+2} + 2e^*_{2k+3} + \ldots + (m+n-2k)e^*_{m+n+1} = z_x.$$

By 5.6.2, $\rho_x = \rho|_{\mathfrak{h}_x}$; since $z_x \in \mathfrak{h}_x$ we get

$$a_x - a = (\rho_x - \rho)(z_x) = 0.$$

Finally, $(\Sigma^x)_{k-s} = (\Sigma_k)^x$ gives $\mathrm{DS}_x(\mathfrak{g}_k) = (\mathfrak{g}_x)_{k-s}$ as required. □

6.3.2. Assume that $0 < s \leq k$ and $s \neq k$ for $t = 0$. Combining Lemma 6.3.1 and 5.10, 6.2.1 we obtain the following diagram

$$\begin{array}{ccc} \mathcal{F}^g_{st}(\mathfrak{g}) & \xrightarrow{\mathrm{Res}} & \mathcal{F}^{pr}_q(\mathfrak{g}_k) \\ \mathrm{DS}_x \downarrow & & \mathrm{DS}_x \downarrow \\ \tilde{\mathcal{F}}^g_{st}(\mathfrak{g}_x) & \xrightarrow{\mathrm{Res}_x} & \tilde{\mathcal{F}}^{pr}_q((\mathfrak{g}_x)_{k-s}) \end{array} \tag{9}$$

where $\mathrm{Res}, \mathrm{Res}_x$ are equivalence of categories.

Let us show that this diagram is commutative. Take $N \in \mathcal{F}^{g;q}(\mathfrak{g})$. Since $z = z_x \in \mathfrak{h}_x$, the spaces N^x and xN are z-stable, so

$$\mathrm{Res}_x(\mathrm{DS}_x(N)) = (\mathrm{DS}_x(N))_a = (N^x)_a/(xN)_a.$$

On the other hand,

$$\mathrm{DS}_x(\mathrm{Res}(N)) = \mathrm{DS}_x(N_a) = (N_a)^x/(xN_a).$$

Since $[x, z] = 0$ one has $(xN)_a = x(N_a)$ and $(N^x)_a = (N_a)^x$. Hence $\mathrm{Res}_x(\mathrm{DS}_x(N)) = \mathrm{DS}_x(\mathrm{Res}(N))$ as required.

6.3.3. The case $t = 0$ and $k = s$

Consider the case $t = 0$ and $k = s > 0$. Note that

$$\mathrm{Res}(N) := \{v \in N |\ zv = \mu(z)v\}, \tag{10}$$

where z is as in 6.3. From the proof of Lemma 6.3.1 we see that $z \in \mathfrak{h}_x$.

Note that $(\mathfrak{g}_x)_{k-s} = 0$, so $\mathcal{F}\mathrm{in}((\mathfrak{g}_x)_{k-s})$ is the category of finite-dimensional super-vector spaces, which we denote by $sVect$. Define

$$\mathrm{Res}_x : \mathcal{F}\mathrm{in}(\mathfrak{g}_x) \to\ sVect$$

by formula (10). Using the arguments of 6.3.2 we obtain the following commutative diagram

$$\begin{array}{ccc} \mathcal{F}^g_{st}(\mathfrak{g}) & \xrightarrow{\text{Res}} & \mathcal{F}^{pr}_q(\mathfrak{g}_k) \\ \text{DS}_x \downarrow & & \text{DS}_x \downarrow \\ \tilde{\mathcal{F}}^g_{st}(\mathfrak{g}_x) & \xrightarrow{\text{Res}_x} & sVect \end{array}$$

By 6.2.1, Res is the equivalence of categories; we will describe Res_x below. Set

$$L' := L_{\mathfrak{g}_x}(\mu).$$

From the proof of Lemma 6.3.1 we see that $z(\alpha) = \{1, 2\}$ for each $\alpha \in \Sigma^x$, so $\text{Res}_x(L') = \mathbb{C}$. If $k = s = m$, then $\tilde{\mathcal{F}}^g_{st}(\mathfrak{g}_x) = \tilde{\mathcal{F}}^g(\mathfrak{g}_x)$ is a semisimple category with $\text{Irr}\big(\tilde{\mathcal{F}}^g(\mathfrak{g}_x)\big) = \{L', \Pi(L')\}$ and the above formula describes Res_x; in particular Res_x is an equivalence of categories.

If $s = k < m$, then $\tilde{\mathcal{F}}^g_{st}(\mathfrak{g}_x) = \tilde{\mathcal{F}}^g(\mathfrak{g}_x)$ is a semisimple category and

$$\text{Irr}\big(\tilde{\mathcal{F}}^g(\mathfrak{g}_x)\big) = \{L', \Pi(L'), (L')^{\sigma_x}, \Pi((L')^{\sigma_x})\}.$$

The eigenvalues of z on $(L')^{\sigma_x} = L_{\mathfrak{g}_x}(\sigma(\mu))$ lie in the set $\sigma_x(\mu)(z) - \mathbb{N}$. Recall that $t = 0$ and μ has the diagram with the sign $+$. Using the notation of the proof of Lemma 6.3.1 we have

$$(\mu - \sigma_x(\mu))(z) = 2(\mu|e^*_{2k+1}) > 0,$$

so $\text{Res}_x\big((L')^{\sigma_x})\big) = 0$. Hence the action of Res_x is

$$\text{Res}_x(L') = \mathbb{C}, \quad \text{Res}_x(\Pi(L')) = \Pi\mathbb{C}, \quad \text{Res}_x((L')^{\sigma_x}) = \text{Res}_x(\Pi((L')^{\sigma_x})) = 0.$$

6.4. Graded multiplicity

Retain notation of 2.3.1. We fix r and denote the graded multiplicity $[\text{DS}_r(L(f)) : L(f')]$ by $\frac{f}{f'}$.

Note that $\frac{f}{f'} = 0$ if $\text{core}(f) \neq \text{core}(f')$ or $\text{atyp}\, f - \text{atyp}\, f' \neq r$ (where atyp stands for the atypicality, i.e. the number of the symbols $\times$ in the diagram).

6.4.1. Case: $\mathfrak{osp}(2n+1|2n)$: switch functor

By [11], Lemma 19 the translation functor $T^{\emptyset}_{\emptyset}$ ("switch functor") acts on simple modules $L(\mu) \in \mathcal{F}^{\emptyset}$ as follows:

$$T^{\emptyset}_{\emptyset}(L(\mu)) = L(\mu^{sw}),$$

where the diagram of μ^{sw} is obtained form the diagram of μ by sign change ($\mu^{sw} = \mu$ if the diagram of μ does not have a sign). Since DS commutes with the translation functors, we get

$$[\mathrm{DS}_r(L(\lambda)) : L(\nu)] = [\mathrm{DS}_r(L(\lambda^{sw})) : L(\nu^{sw})].$$

6.4.2. Let $\sigma(f)$ be the diagram obtained from f by the change of sign (for $\mathfrak{osp}(2m|2n)$ this notation was used before).

6.4.3. **Corollary**. *Let f, f' be weight diagrams with* $\mathrm{core}(f) = \mathrm{core}(f')$. *One has*

(i) $\frac{\sigma(f')}{\sigma(f)} = \frac{f'}{f}$ *except for* $t = 0$, $\mathrm{howl}(f) = \emptyset$;
(ii) $\frac{f'}{f} = \frac{\mathrm{howl}(f')}{\mathrm{howl}(f)}$.

Proof. For $t = 1$ the formula $\frac{\sigma(f')}{\sigma(f)} = \frac{f'}{f}$ follows from 6.4.1 and for $t = 2$ one has $\sigma(f) = f$, $\sigma(f') = f'$. Lemma 5.7 (i) gives $\frac{\sigma(f')}{\sigma(f)} = \frac{f'}{f}$ for $t = 0$ except for the case $\mathrm{howl}(f) = \emptyset$.

Combining 6.3.2 and Corollary 5.3 we obtain $\frac{f'}{f} = \frac{\mathrm{howl}(f')}{\mathrm{howl}(f)}$ for all cases except for $t = 0$, $\mathrm{rank}\, x = \mathrm{atyp}(f') =: r$. In the remaining case f' is an $\mathfrak{osp}(2m|2n)$-diagram and f is a typical $\mathfrak{osp}(2(m-r)|2(n-r))$-diagram. By 6.3.3, $\frac{f'}{f} = \frac{\mathrm{howl}(f')}{\mathrm{howl}(f)}$ if $r = m$ or if f has the sign $+$. Consider the remaining case when $r < m$ and f has the sign $-$ (f has a sign, since it is typical with $t = 0$). By Lemma 5.7 (i) one has $\frac{\sigma(f')}{\sigma(f)} = \frac{f'}{f}$ (since $r < m$). Since $\sigma(f)$ has sign $+$ we have $\frac{\sigma(f')}{\sigma(f)} = \frac{\mathrm{howl}(\sigma(f'))}{\mathrm{howl}(\sigma(f))}$ and the required formula $\frac{f'}{f} = \frac{\mathrm{howl}(f')}{\mathrm{howl}(f)}$ follows from the fact that σ commutes with howl for $t = 0$. $\square$

6.4.4. Remark

Theorems 8.2, 9.3 imply $\frac{\sigma(f')}{\sigma(f)} = \frac{f'}{f}$ in the remaining cases.

7. Recursive formulae for $[\mathrm{DS}_x(L(\lambda)) : L_{\mathfrak{g}_x}(\nu)]$

Using that *DS* commutes with translation functors, we establish recursive formulas for the multiplicities $[\mathrm{DS}_x(L(\lambda)) : L_{\mathfrak{g}_x}(\nu)]$. This will ultimately allow us to reduce the computation of the multiplicities for DS_1 to the case $\mathfrak{g} = \mathfrak{osp}(2+t|2)$. For a similar reduction in the $\mathfrak{gl}(m|n)$-case see [14].

We fix r and retain the notation of 6.4.

7.1. Translation functors

Recall that DS commutes with the translation functors. Let g_0, g_1 be two core diagrams. Consider the translation functors

$$T_{g_1}^{g_0} : \tilde{\mathcal{F}}^{g_1}(\mathfrak{g}) \to \tilde{\mathcal{F}}^{g_0}(\mathfrak{g}), \quad T_{g_1}^{g_0} : \tilde{\mathcal{F}}^{g_1}(\mathfrak{g}_x) \to \tilde{\mathcal{F}}^{g_0}(\mathfrak{g}_x).$$

Since translation functors are exact, they induce morphisms on the Grothendieck ring. For $N \in \mathcal{F}^{g_1}(\mathfrak{g})$ and $L' \in \mathrm{Irr}(\mathfrak{g}_x)^{g_0}$ we obtain

$$\begin{aligned}[\mathrm{DS}_r(T_{g_1}^{g_0}(N)) : L'] &= [T_{g_1}^{g_0}(DS_r(N)) : L'] \\ &= \sum_{L_1 \in \mathrm{Irr}(\mathfrak{g}_x)^{g_1}} [DS_r(N) : L_1][T_{g_1}^{g_0}(L_1) : L'].\end{aligned} \tag{11}$$

7.1.1. Assume that the number of core symbols in g_0 is larger than the number of core symbols in g_1, i.e. the atypicality of g_1 is larger than the atypicality of g_0. We will use the following results of [12], Lemmatta 7, 13, 14:

the image of a simple module $L \in \mathrm{Irr}(\mathfrak{g}_x)^{g_1}$ is either zero or simple;

for $L_1, L_2 \in \mathrm{Irr}(\mathfrak{g}_x)^{g_1}$ with $T_{g_1}^{g_0}(L_1) \cong T_{g_1}^{g_0}(L_2) \neq 0$ one has $L_1 \cong L_2$.

7.1.2. These results will be sufficient for us. A complete description of translation functors on irreducible modules and projective covers can be obtained from [4], [5], [6].

7.1.3. Take g_0, g_1 satisfying the assumption in 7.1.1. For each $L' \in \mathrm{Irr}(\mathfrak{g}_x)^{g_0}$ there exists at most one (up to isomorphism) $L_1 \in \mathrm{Irr}(\mathfrak{g}_x)^{g_1}$ such that $T_{g_1}^{g_0}(L_1) \cong L'$. Then (11) gives for $N \in \mathcal{F}^{g_1}(\mathfrak{g})$

$$[\mathrm{DS}_r(T_{g_1}^{g_0}(N)) : L'] = [T_{g_1}^{g_0}(DS_r(N)) : L'] = [DS_r(N) : L_1].$$

In particular,

$$[DS_r(N) : L_1] \neq 0 \implies T_{g_1}^{g_0}(N) \neq 0. \tag{12}$$

Now take $N := L(\lambda)$ and ν such that $[\mathrm{DS}_r(L(\lambda)) : L_{\mathfrak{g}_x}(\nu)] \neq 0$ and $T_{g_1}^{g_0}(L_{\mathfrak{g}_x}(\nu)) \neq 0$. Then, by 7.1.1,

$$T_{g_1}^{g_0}(L_{\mathfrak{g}_x}(\nu)) = L_{\mathfrak{g}_x}(\nu_1)$$

for some ν_1 and $T_{g_1}^{g_0}(L(\lambda))$ is either zero or simple. By above, $T_{g_1}^{g_0}(L(\lambda)) \neq 0$, so

$$T_{g_1}^{g_0}(L(\lambda)) = L(\lambda_1)$$

for some λ_1. Using (11) and Corollary 6.4.3 we conclude

$$\frac{f(\lambda)}{f(\nu)} = \frac{\mathrm{howl}(f(\lambda_1))}{\mathrm{howl}(f(\nu_1))}, \tag{13}$$

where $f(\lambda)$ stands for the weight diagram of λ.

7.2. Translation functors T_u

We describe some translation functors via their effect (called *elementary change* in [12], Section 6.3) on core/weight diagrams. Let a be a non-negative integer. For each diagram f we denote by $\text{pos}_a(f)$ the subdiagram corresponding to the positions $a, a+1$. For a core diagram g_1 with $\text{pos}_a(g_1) = \circ\circ$ we denote by $\phi'_a(g_1)$ the core diagram obtained from g_1 by changing $\text{pos}_a(g_1) = \circ\circ$ to $\text{pos}_a(g) =><$; for instance,

$$\phi'_1(< \circ\circ >) = <><> .$$

We denote by T_a the functor which acts as $T_{g_1}^{\phi'_a(g_1)}$ on $\tilde{\mathcal{F}}^{g_1}$ with $\text{pos}_a(g_1) = \circ\circ$ and by zero on $\tilde{\mathcal{F}}^{g'}$ with $\text{pos}_a(g') \neq \circ\circ$. Note that T_a reduces the atypicality by 1.

7.2.1. Take $u > 0$. For a diagram f with $\text{pos}_u(f) = \times\circ$ we define the diagram $\phi_u(f)$ obtained from f by changing $\times\circ$ in the positions $(u, u+1)$ to $><$ (i.e., f and $\phi_u(f)$ have the same signs and the same symbols in all positions except $u, u+1$ and $\text{pos}_u(\phi_u(f)) =><$). For instance, $\phi_1(\times\times\times\circ)$ is not defined and $\phi_2(\times\times\times\circ) = \times\times ><$.

7.2.2. By [12], for $u > 0$ one has

$$T_u(L(\mu)) = L(\phi_u(\mu))$$

if $\phi_u(\mu)$ is defined and $T_u(L(\mu)) = 0$ otherwise.

7.3. **Corollary**. *For $u > 0$ one has*

$$\frac{f'_u \times \times f'}{f_u \times \circ f} = \frac{f'_u \circ \times f'}{f_u \times \circ f} = \frac{f'_u \circ \circ f'}{f_u \times \circ f} = 0$$

(with all possible signs) and

$$\frac{f'_u \times \circ f'}{f_u \times \circ f} = \frac{f'_u f'}{f_u f}$$

where f_u, f'_u stands for the subdiagrams corresponding to the positions $0, 1, \ldots, u-1$.

Proof. Recall from (12) that for irreducible $N = L(\lambda)$

$$[\text{DS}_r(T_{g_1}^{g_0}(L(\lambda)) : T_{g_1}^{g_0}(L_1)] = [DS_r(L(\lambda) : L_1].$$

Take $u > 0$. Take $\tilde{f} = f_u \times \circ f$ (i.e., $\text{pos}_u(\tilde{f}) = \times\circ$). Combining (12) and 7.2.1 we get $\frac{\tilde{f}'}{\tilde{f}} = 0$ if $\text{pos}_u(\tilde{f}') \neq \times\circ$; this gives the first formula. If $\tilde{f}' = f'_u \times \circ f'$, then T_u transforms $\tilde{f}'$ to $f'_u >< f'$. Combining 7.2.1 and (13) we have

$$\frac{f'_u \times \circ f'}{f_u \times \circ f} = \frac{\mathrm{howl}(f'_u >< f')}{\mathrm{howl}(f_u >< f)}.$$

Using $\mathrm{howl}(f_u >< f) = \mathrm{howl}(f_u f)$ we obtain the second formula. □

7.4. Reduction to the case $\nu = 0$

Let f be a weight diagram and u be the coordinate of the rightmost symbol $\times$ in f. Using 6.4.3 (ii) and 7.3 we reduce the computation of $\frac{f'}{f}$ to the situation when f, f' are core-free and $u = 0$; in the light of 6.4.3 (i) we can assume that f has sign $-$ for $t = 1$ (for $t = 0, 2$ the diagram f does not have sign if $u = 0$). Notice that in this case $\lambda(f') = 0$, i.e. we reduced the problem to the computation of $[\mathrm{DS}_r(L(\lambda)) : L_{\mathfrak{g}_x}(0)]$ for $\mathfrak{g} = \mathfrak{osp}(2n + t|2n)$.

For $\mathfrak{gl}(m|n)$ a similar reasoning reduces the computation of $[\mathrm{DS}_r(L(\lambda)) : L(\nu)]$ to the case when ν has the empty diagram, i.e. to the case when $\mathfrak{g}_x = 0$ (see also [14]). In the $\mathfrak{osp}$-case this is done in Corollaries 7.5.2, 7.6.4 below.

7.5. The case $\mathfrak{osp}(2m + 1|2n)$

Consider the case $\mathfrak{g} = \mathfrak{osp}(2m + 1|2n)$. We assume, as always, that "the signs disappear" if the zero position is empty, i.e. $\pm \times^{i-1} f$ stands for $\circ f$ for $i = 1$.

7.5.1. By [12], for the translation functor $T_0 := T_{\emptyset}^{><}$ one has $T_0(L(\mu)) \neq 0$ if and only if the diagram of μ has the sign $+$ and $\mathrm{pos}_0(\mu) = \times^i \circ$ for $i > 0$; moreover,

$$T_0(L(+\times^i \circ f) = L(\overset{\times^{i-1}}{>} < f).$$

Using (12) we obtain for $j \geq 1$

$$\frac{\tilde{f}}{+\times^j \circ f} \neq 0 \implies \tilde{f} = +\times^p \circ f' \ \text{ for some } p \geq 1.$$

7.5.2. **Lemma**. *For $p, i \geq 1$ one has*

$$\frac{+\times^p \circ\circ f}{+\times^i} = \frac{-\times^{p-1} f}{-\times^{i-1}} \qquad \frac{+\times^p \circ \times f}{+\times^i} = \frac{+\times^p f}{-\times^{i-1}}$$

Proof. Using (13) for T_0 we obtain for $* \in \{\circ, \times\}$

$$\frac{+\times^p \circ * f}{+\times^i} = \frac{\overset{\times^{p-1}}{>} < * f}{\overset{\times^{i-1}}{>} <}.$$

One has

$$\mathrm{howl}(\overset{\times^{p-1}}{>}<*f)=\begin{cases}-\times^{p-1}f & \text{if } *=\circ\\ +\times^{p}f & \text{if } *=\times\end{cases}$$

and $\mathrm{howl}(\overset{\times^{i-1}}{>}<)=-\times^{i-1}$. Now (13) gives

$$\frac{+\times^p\circ\circ f}{+\times^i}=\frac{-\times^{p-1}f}{-\times^{i-1}}\qquad\frac{+\times^p\circ\times f}{+\times^i}=\frac{+\times^p f}{-\times^{i-1}}$$

as required. □

7.5.3. **Corollary**. *Take $i\geq 1$. One has*

$$\frac{-\times^p\underbrace{\circ\ldots\circ}_{2i-1\ times}\times f}{-\times^i}=\frac{-\times^{p-i+1}f}{\emptyset}\qquad\frac{-\times^p\underbrace{\circ\ldots\circ}_{j\ times}f}{-\times^i}=\frac{-\times^{p-i}\underbrace{\circ\ldots\circ}_{j-2i\ times}f}{\emptyset}\quad for\ \ j\geq 2i.$$

Moreover, $\frac{\tilde f}{-\times^i}=0$ if $\tilde f$ is not as above, i.e. $\tilde f\neq-\times^p\underbrace{\circ\ldots\circ}_{j\ times}f$ for some $p\geq i$ and $j\geq 2i-1$.

Proof. The statement follows by induction from Lemma 7.5.2 and Corollary 6.4.3. □

7.6. The case $\mathfrak{osp}(2m|2n)$

Recall that the simple $OSp(2m|2n)$-modules are in one-to-one correspondence with the unsigned diagram (see 3.1.8). For a non-empty diagram f we will use the sign $\circ f$ for $+\circ f\oplus-\circ f$, i.e. $L(\circ f)$ is a simple $OSp(2m|2n)$-module which is the direct sum of $\mathfrak{osp}(2m|2n)$-modules $L(+\circ f)$ and $L(-\circ f)$.

For an empty diagram we have $L(\emptyset)=\mathbb{C}$; notice that

$$\mathrm{howl}(\pm\circ>)=\emptyset;\qquad\mathrm{howl}(\circ>)\neq\emptyset.$$

7.6.1. For the translation functor $T_0:=T_{\emptyset}^{><}$ one has $T_0(L(\mu))\neq 0$ if and only if $\mathrm{pos}_0(\mu)=\times^i\circ$ for $i>0$ and

$$T_0(L(\times^i\circ f))=L(\overset{\times^{i-1}}{>}<f).$$

Using 7.1.3 we obtain for $j\geq 1$

$$\frac{\tilde f}{\times^j\circ f}\neq 0\implies\tilde f=\times^p\circ f'\ \text{ for some } p>0. \tag{14}$$

7.6.2. The translation functor $T_{>}^{\circ>}$ is given by

$$T_{>}^{\circ>}(L(\overset{\times^p}{>}\circ f) = L(\times^p > f), \quad T_{>}^{\circ>}(L(\overset{\times^p}{>}\times f) = L(\times^{p+1} > f)$$

for each $p \geq 0$ (where, $L(\circ f) = L(+\circ f) \oplus L(-\circ f)$).

7.6.3. The translation functor $T_{\circ>}^{>}$ is adjoint to the functor $T_{>}^{\circ>}$. It is given by

$$T_{\circ>}^{>}(L(\pm\circ > f) = L(>\circ f), \quad T_{\circ>}^{>}(L(\times^p > f)) = L(\overset{\times^p}{>}\circ f) \oplus L(\overset{\times^{p-1}}{>}\times f)$$

for each $p > 0$ (there is a misprint in [12]).

7.6.4. **Lemma**. *Take $i, p \geq 1$.*

(i) $\frac{\times^p\circ f}{\times^i} = \frac{\overset{\times^{p-1}}{>}\ f}{\overset{\times^{i-1}}{>}} \qquad \frac{\overset{\times^p}{>}\circ f}{\overset{\times^i}{>}} = \frac{\times^p f}{\times^i}$.

(ii) $\frac{\overset{\times^{p-1}}{>}\times f}{\overset{\times^i}{>}} = 0$.

(iii) $\frac{\overset{\times^p}{>}\circ f}{>} = \frac{\overset{\times^{p-1}}{>}\times f}{>} = \frac{\times^p f}{\emptyset}$.

Proof. Using (13) for T_0 we obtain

$$\frac{\times^p\circ f}{\times^i} = \frac{\overset{\times^{p-1}}{>}<f}{\overset{\times^{i-1}}{>}\ <} = \frac{\overset{\times^{p-1}}{>}\ f}{\overset{\times^{i-1}}{>}} \tag{15}$$

which establishes the first formula. Using 7.1 for $T_{>}^{\circ>}$ we obtain

$$\begin{aligned}
\frac{\times^p f}{\times^i} &= \frac{\times^p > f}{\times^i >} = [\mathrm{DS}_r(T_{>}^{\circ>}(L(\overset{\times^p}{>}\circ f)) : L(\times^i >)] \\
&= \sum[\mathrm{DS}_r(L(\overset{\times^p}{>}\circ f)) : L_1][T_{>}^{\circ>}(L_1) : L(\times^i >)] \\
&= [\mathrm{DS}_r(L(\overset{\times^p}{>}\circ f)) : L(\overset{\times^i}{>})] + [\mathrm{DS}_r(L(\overset{\times^p}{>}\circ f)) : L(\overset{\times^{i-1}}{>}\times)] \\
&= \frac{\overset{\times^p}{>}\circ f}{\overset{\times^i}{>}} + \frac{\overset{\times^p}{>}\circ f}{\overset{\times^{i-1}}{>}\ \times}.
\end{aligned}$$

By Corollary 7.3 the second summand in the last formula is zero; this implies the second formula. Similarly, $T_{>}^{\circ>}(L(\overset{\times^{p-1}}{>}\times f)) = L(\times^p > f)$ implies

$$\frac{\times^p f}{\times^i} = \frac{\overset{\times^{p-1}}{>}\times f}{\overset{\times^i}{>}} + \frac{\overset{\times^{p-1}}{>}\times f}{\overset{\times^{i-1}}{>}\ \times}.$$

By (14) if $f = \times f'$, then $\frac{\times^p f}{\times^i} = 0$, so the both summands in the right-hand side are equal to 0; in particular, $\frac{\overset{\times^{p-1}}{>}\times\times f'}{\overset{\times^i}{>}} = 0$.

If $f = \circ f'$ we have

$$\frac{\times^p \circ f'}{\times^i} = \frac{\overset{\times^{p-1}}{>} \times \circ f'}{\overset{\times^i}{>}} + \frac{\overset{\times^{p-1}}{>} \times \circ f'}{\overset{\times^{i-1}}{>} \times} = \frac{\overset{\times^{p-1}}{>} \times \circ f'}{\overset{\times^i}{>}} + \frac{\overset{\times^{p-1}}{>} f'}{\overset{\times^{i-1}}{>}}.$$

Using (i) we conclude

$$\frac{\overset{\times^{p-1}}{>} \times \circ f'}{\overset{\times^i}{>}} = 0.$$

This establishes (ii).

Using 7.1 for $T_{>}^{\circ >}$ we obtain

$$\begin{aligned} \frac{\times^p f}{\emptyset} &= \frac{\times^p > f}{+\circ >} = [\mathrm{DS}_r(T_{>}^{\circ >}(L(\overset{\times^p}{>} \circ f)) : L(+\circ >)] \\ &= \sum [\mathrm{DS}_r(L(\overset{\times^p}{>} \circ f)) : L_1][T_{>}^{\circ >}(L_1) : L(+\circ >)] \\ &= [\mathrm{DS}_r(L(\overset{\times^p}{>} \circ f)) : L(>)] = \frac{\overset{\times^p}{>} \circ f}{>} \end{aligned}$$

and, similarly,

$$\frac{\times^p f}{\emptyset} = \frac{\overset{\times^{p-1}}{>} \times f}{>}.$$

This establishes (iii). □

7.6.5. **Corollary**. *Take $i \geq 1$. One has $\frac{\times^p \times f}{\times^i} = 0$ and $\frac{\times^p \circ f}{\times^i} = \frac{\overset{\times^{p-1}}{>} f}{\overset{\times^{i-1}}{>}}$. Moreover,*

$$\frac{\overset{\times^p}{>} \underbrace{\circ \ldots \circ}_{j\ times} f'}{\overset{\times^i}{>}} = \frac{\overset{\times^{p-i}}{>} \underbrace{\circ \ldots \circ}_{j-2i\ times} f'}{>}$$

and $\frac{\overset{\times^p}{>} f}{\overset{\times^i}{>}} \neq 0$ implies that $p \geq i$ and the diagram f is as above (i.e., $f = \underbrace{\circ \ldots \circ}_{j\ times} f'$ for some $j \geq 2i$).

8. Computation of $\mathrm{DS}_1(L)$ in terms of arc diagrams

8.1. Arc diagrams

We assign to each core-free diagram an *arc diagram* $Arc(f)$; if f is contains at most one $\times$ at the zero position ("$\mathfrak{gl}$-type"), the corresponding arc diagram coincides with the usual arc diagram introduced in [12]. It differs from the arc- or cup diagrams of [4]. Advantages of our weight and arc diagrams are that they can immediately be read off from the weight λ and describe the effect of DS very nicely. On the other they do not connect to Khovanov algebras of type D and therefore to the Kazhdan-Lusztig combinatoric of parabolic category $\mathcal{O}$.

An *arc diagram* is the following data: a diagram f, where the symbols $\times$ at the zero position are drawn vertically and a collection of non-intersecting arcs. Each arc connects one symbol $\times$ (the left end) with one or two empty symbols according to the following rules:

if the symbol $\times$ has non-zero coordinate, the arc connects this symbol with one empty symbol;

for $\ell = 0$ (i.e., $t = 0, 1$) the lowest symbol $\times$ in the zero position, is connected by an arc with one empty symbol and the other symbols $\times$ in the zero position are connected with two empty symbols;

for $\ell = 1$ (i.e., $t = 2$) all symbols $\times$ in the zero position are connected with two empty symbols.

An empty position in f is called *free* in the arc diagram if this position is not an end of an arc.

We say that an arc is supported by a symbol $\times$ if this symbol is the left end of the arc; if the arc is supported by a symbol $\times$ with the coordinate a we denote this arc by $arc(a;b)$ (resp., $arc(0;b_1,b_2)$) where b (resp., $b_1 < b_2$) is the coordinate of the right end (resp., right ends) of the arc.

We remark that we can similarly define an arc diagram for any weight diagram by just fixing and ignoring the core symbols.

8.1.1. Partial order

We consider a partial order on the set of arcs by saying that one arc is smaller than another one if the first one is "below" the second one, that is

$arc(a;b) > arc(a';b')$ if and only if $a < a' < b' < b$;

$arc(0;b_1,b_2) > arc(a';b')$ if and only if $b' < b_2$;

$arc(0;b_1,b_2) > arc(0;b'_1,b'_2)$ if and only if $b'_2 < b_2$.

Since the arcs do not intersect, one has

$$\begin{aligned} arc(a;b) > arc(a';b') &\iff a < a' < b \\ arc(0;b_1,b_2) > arc(a';b') &\iff a' < b_2 \end{aligned}$$

and any two distinct arcs of the form $arc(0;b_1,b_2), arc(0;b'_1,b'_2)$ are comparable: either $arc(0;b_1,b_2) > arc(0;b'_1,b'_2)$ or $arc(0;b_1,b_2) < arc(0;b'_1,b'_2)$.

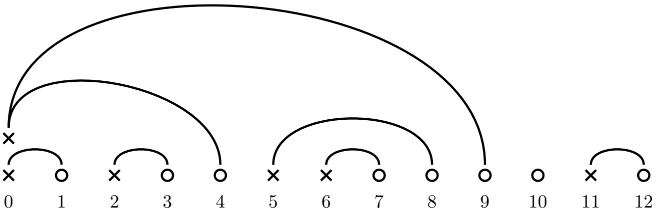
8.1.2. Definition

We assign to a core-free diagram f the arc diagram $Arc(f)$ with the following properties:

each symbol $\times$ is the left end of exactly one arc;
there are no free positions under the arcs.

8.1.3. Example

The weight diagram below does not have a sign for $t = 0$ ($\mathfrak{g} = \mathfrak{osp}(12|12)$) or have one of the signs $\pm$ for $t = 1$ ($\mathfrak{g} = \mathfrak{osp}(13|12)$).

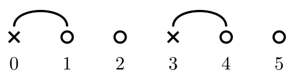


Arc diagram for $\times^2 \circ \times \circ \circ \times \times \circ \circ \circ \circ \times \circ$

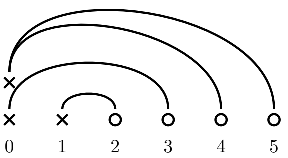
8.1.4. Examples

In the following diagrams we ignore signs. Note that in case the arc connects $\times$ with two empty symbols, this still counts as one arc (also for Theorem 8.2).

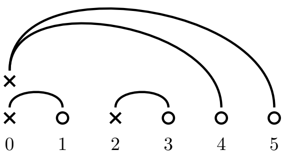
For $\mathfrak{osp}(5|4)$ the arc diagram of $+ \times \circ \circ \times$ is given by



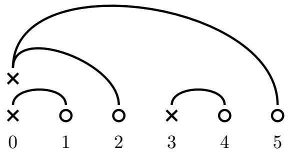
The arc diagrams of $- \times^2 \underbrace{\circ \ldots \circ}_{j \text{ times}} \times$ are the following:



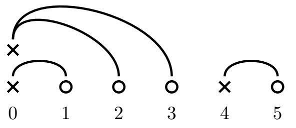
The arc diagram of $- \times^2 \times \circ$



The arc diagram of $- \times^2 \circ \times \circ$

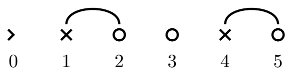


The arc diagram of $-\times^2 \circ \circ \times \circ$

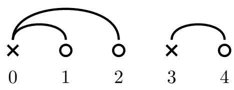


The arc diagram of $-\times^2 \circ \circ\ \circ \times \circ$

For $\mathfrak{osp}(6|4)$ the arc diagram of $> \times \circ \circ \times$ is



and the arc diagram of $\times \circ \circ \times$ is



8.1.5. Description

The arc diagram $Arc(f)$ is unique. For instance, for $\ell = 0$ the arc diagram $Arc(f)$ can be constructed as follows:

first we consider the diagram f' obtained from f by removing all but one symbol $\times$ from the zero position ($f' = f$ if the zero position contains $\circ$ or $\times$);

for f' we construct the arc diagram in the usual way (we connect each symbol $\times$ with the next free empty symbol starting from the rightmost symbol $\times$ and going to the left);

if the zero in f is occupied by $\times^p$ with $p > 1$, we connect each of the remaining $p-1$ symbol $\times$ with next two free positions starting from the lowest symbol $\times$ and going up.

8.1.6. Maximal arcs

Note that an arc $arc(0; b)$ (resp., $(0; b_1, b_2)$) is maximal if and only if it is supported by the top symbol $\times$ in the zero position. Note that $Arc(\times^p f)$ is obtained from $Arc(\times^{p+1} f)$ by "removal" of the maximal arc supported by the top symbol $\times$ in the zero position. Notice that for $\mathfrak{osp}(2n+1|2n)$, when we erase a symbol $\times$ from a signed diagram we either obtain a signed diagram with the same sign or an unsigned diagram if the resulting diagram does not have $\times$ in the zero position.

Consider a diagram of the form $f_1 \times f_2$ where the symbol $\times$ occupies a position $a > 0$. This symbol $\times$ supports a maximal arc $arc(a; b)$ if and only if $Arc(f_1 \circ f_2)$ is obtained from $Arc(f_1 \times f_2)$ by "removal" of the arc $arc(a; 0)$, i.e.

$$Arc(f_1 \circ f_2) = Arc(f_1) \circ Arc(f_2)$$

(if we remove an arc which is not maximal, the resulting diagram is not an arc diagram).

For instance, in Example 8.1.3, the maximal arcs are $arc(11; 12)$ and $arc(0; 4, 9)$.

8.2. **Theorem**. *(i) The module $L(\nu)$ is a subquotient of $\mathrm{DS}_1(L(\lambda))$ if and only if $Arc(\mathrm{howl}(\nu))$ is obtained from $Arc(\mathrm{howl}(\lambda))$ by removing a maximal arc and, in addition, in the $\mathfrak{osp}(2m+1|2n)$-case, if ν has a sign, then the signs of λ and ν are equal.*

(ii) Let e be the number of free positions in $Arc(\mathrm{howl}(\lambda))$ which are to the left of the maximal arc.

For $\mathfrak{osp}(2m+1|2n)$ one has

$$[\mathrm{DS}_1(L(\lambda)) : L(\nu)] = \begin{cases} (1|0) & \text{if } e = 0; \\ (2|0) & \text{if } e \text{ is even and } e \neq 0; \\ (0|2) & \text{if } e \text{ is odd.} \end{cases} \tag{16}$$

The same formula holds for the case $t = 2$. For $t = 0$

$$[\mathrm{DS}_1(L(\lambda)) : L(\nu)] = \begin{cases} (1|0) & \text{if } e \text{ is even;} \\ (0|1) & \text{if } e \text{ is odd.} \end{cases} \tag{17}$$

8.2.1. Examples

Take $r = 1$ and use the notations of Section 7.

For $\mathfrak{osp}(5|4)$ we have

$$\frac{+\times\circ\circ\times}{\circ\circ\circ\times} = (1|0) \qquad \frac{+\times\circ\circ\times}{+\times} = (0|2)$$

and $\frac{+\times\circ\circ\times}{f'} = 0$ in other cases. The multiplicity is

$$\frac{-\times^2 \underbrace{\circ\ldots\circ}_{j \text{ times}} \times}{-\times^2} = \begin{cases} 0 & \text{for } j < 3, \\ (1|0) & \text{for } j = 3, \\ (2|0) & \text{if } j > 3 \text{ is odd,} \\ (0|2) & \text{if } j > 3 \text{ is even.} \end{cases}$$

For $\mathfrak{osp}(4|4)$ we have

$$\frac{+\circ\times\circ\times}{+\circ\times} = \frac{+\circ\times\circ\times}{-\circ\times} = (0|1).$$

For $\mathfrak{osp}(6|4)$ we have

$$\frac{> \times \circ \circ \times}{> \circ \circ \circ \times} = (1|0) \qquad \frac{+ \times \circ \circ \times}{+ \times} = (0|2)$$

and $\frac{+\times\circ\circ\times}{f'} = 0$ in other cases.

8.3. Low rank cases

The proof will be a reduction to the cases $\mathfrak{osp}(2+t|2)$ for $t = 0, 1, 2$. For $t = 0, 1$ $\mathfrak{g}_x = 0$, so $\mathfrak{g}_x$-modules are supervector spaces. For $t = 2$ $\mathfrak{g}_x = \mathbb{C}$. For the principal block $\mathcal{F}^{>}(\mathfrak{g})$ $DS_x(\mathcal{F}^{>}(\mathfrak{g}))$ is the category of finite-dimensional supervector spaces with trivial action of $\mathfrak{g}_x$. The simple modules in the principal block are of the form $L(\lambda_j)$, $j \in \mathbb{Z}$ for $t = 0$, and $j \in \mathbb{N}$ for $t = 1, 2$. For $t = 0$

$$\mathrm{DS}_1(L(\lambda_j)) \cong \Pi^j(\mathbb{C}) \quad \forall j \in \mathbb{Z}$$

where $\lambda_j := j\varepsilon_1 + |j|\delta_1$ for $j \in \mathbb{Z}$. For $t = 1$ we put $\lambda_0 := 0$ and $\lambda_j := j\varepsilon_1 + (j-1)\delta_1$ for $j \geq 1$. For $t = 2$ we put $\lambda_j := j\varepsilon_1 + j\delta_1$ for $j \geq 0$. Then one has the uniform rule [9]

$$\begin{aligned} &\mathrm{DS}_1(L(\lambda_0) = \mathrm{DS}_1(L(\lambda_1)) \cong \mathbb{C} \text{ and} \\ &\qquad\qquad \mathrm{DS}_1(L(\lambda_j)) \cong \Pi^{j-1}(\mathbb{C})^{\oplus 2} \text{ for } j \geq 2. \end{aligned}$$

8.4. Proof

Set $f := \mathrm{howl}(\nu)$. Denote by u the coordinate of the rightmost symbol $\times$ in f.

If $u > 0$, then $Arc(f)$ has a minimal arc $arc(u; u+1)$. If $\frac{f'}{f} \neq 0$, then, by 7.3, f' has a minimal arc $arc(u; u+1)$ and

$$\frac{f'}{f} = \frac{\overline{f}'}{\overline{f}},$$

where $\overline{f}'$ (resp., $\overline{f}$) are obtained from f' (resp., f) by "shrinking" the minimal arc $arc(u; u+1)$. Clearly, $Arc(f)$ can be obtained from $Arc(f')$ by deleting a maximal arc if and only if $Arc(\overline{f})$ can be obtained from $Arc(\overline{f}')$ by deleting a maximal arc. Each maximal arc in $Arc(\overline{f})$ corresponds to a maximal arc in $Arc(f)$ and the number of free positions to the left of each maximal arc is the same. This reduces the statement to the case $u = 0$. Corollaries 7.5.3, 7.6.5 reduce the statement to the cases when $f = \emptyset$ or $f =>$. Since $r = 1$, the case $f = \emptyset$ correspond to $\mathfrak{osp}(2|2), \mathfrak{osp}(3|2)$ and the case $f =>$ correspond to the case $\mathfrak{osp}(4|2)$. For these cases the formula was checked in [9] (see 8.3). □

We will use the following lemma.

8.4.1. **Lemma**. *Take τ as in 3.7.7. For $r = 1$ one has $\frac{f}{f'} = \frac{\tau(f)}{\tau(f')}$.*

Proof. One has to check the following cases

$$\begin{array}{ll} f = \overset{\times^p}{>} \circ f_1 \times f_2 & f' = \overset{\times^p}{>} \circ f_1 \circ f_2 \\ f = \overset{\times^p}{>} \times f_1 \times f_2 & f' = \overset{\times^p}{>} \times f_1 \circ f_2 \\ f = \overset{\times^p}{>} \times f_1 & f' = \overset{\times^p}{>} \circ f_1 \\ f = \overset{\times^{p+1}}{>} \circ f_1 & f' = \overset{\times^p}{>} \circ f_1 \\ f = \overset{\times^{p+1}}{>} \times f_1 & f' = \overset{\times^p}{>} \times f_1. \end{array}$$

This can be easily done using the properties of maximal arcs discussed in 8.1.6. □

9. Semisimplicity of $\mathrm{DS}_x(L)$

Retain the notation of 3.7.8. Denote by $\mathcal{F}_+(\mathfrak{g})$ the Serre subcategory of $\tilde{\mathcal{F}}(\mathfrak{g})$ generated by $L \in \mathrm{Irr}\big(\tilde{\mathcal{F}}(\mathfrak{g})\big)$ with $\mathrm{dex}(L) = 1$. By Corollary 6.2.2, $\mathcal{F}_+(\mathfrak{g})$ is semisimple.

In Theorem 9.3 we will show that $\mathrm{DS}_x(\mathcal{F}_+(\mathfrak{g})) \subset \mathcal{F}_+(\mathfrak{g}_x)$. As a result, $\mathrm{DS}_x(L)$ is semisimple for each simple finite-dimensional module L.

9.1. Supercharacters

Recall that besides the usual character we have the supercharacter

$$\mathrm{sch}\, L(\lambda) = (-1)^{p(\lambda)}\pi(\mathrm{ch}\, L(\lambda)),$$

where $\pi : \mathbb{Z}[\Lambda_{m|n}] \to \mathbb{Z}[\Lambda_{m|n}]$ is the linear involution given by

$$\pi(e^\mu) := (-1)^{p(\mu)} e^\mu.$$

The supercharacter ring of $\mathcal{F}\mathrm{in}$ is the image of the map $\mathrm{sch} : \mathcal{F}\mathrm{in} \to \mathbb{Z}[\mathfrak{h}^*]$; we denote this ring by $\mathcal{J}(\mathfrak{g})$.

9.2. We will use the following lemma.

Lemma. *Let $M \in \mathcal{F}\mathrm{in}(\mathfrak{g})$ and $N \in \mathcal{F}_+(\mathfrak{g})$ be such that $\mathrm{sch}\, M = \mathrm{sch}\, N$ in the supercharacter ring. If $\dim M \le \dim N$, then $M \cong N$.*

Proof. For each $\nu \in \Lambda^{(t)}_{m+\ell|n}$ we set

$$(d_0(\nu)|d_1(\nu)) := [N : L(\nu)], \quad (d'_0(\nu)|d'_1(\nu)) := [M : L(\nu)].$$

Since $N \in \mathcal{F}_+(\mathfrak{g})$ we have $d_0(\nu)d_1(\nu) = 0$.

Since $\{\operatorname{sch} L(\nu)|\ \nu \in \Lambda^{(t)}_{m+\ell|n}\}$ are linearly independent, the equality $\operatorname{sch} M = \operatorname{sch} N$ implies

$$(d_0'(\nu)|d_1'(\nu)) = (d_0(\nu) + j(\nu)|d_1(\nu) + j(\nu)) \quad \text{for some } j(\nu) \in \mathbb{Z}.$$

Combining $d_0(\nu)d_1(\nu) = 0$ with $d_0'(\nu), d_1'(\nu) \geq 0$, we obtain $j(\nu) \geq 0$ for each ν. Using $\dim M \leq \dim N$ we get $j(\nu) = 0$ for each ν, that is

$$\forall\nu \quad [M : L(\nu)] = [N : L(\nu)]. \tag{18}$$

Hence $M \in \mathcal{F}_+(\mathfrak{g})$. Since $\mathcal{F}_+(\mathfrak{g})$ is semisimple, M and N are completely reducible. Thus (18) gives $M \cong N$. $\square$

9.3. **Theorem.**

(i) *For each x one has $\operatorname{DS}_x\big(\mathcal{F}_+(\mathfrak{g})\big) = \mathcal{F}_+(\mathfrak{g}_x)$. In particular $\operatorname{DS}_x(L(\lambda))$ is semisimple for any x and any simple finite-dimensional module L.*

(ii) *For each simple finite-dimensional module L one has $\operatorname{DS}_{r+1}(L) \cong \operatorname{DS}_1(\operatorname{DS}_r(L))$.*

Proof. It is enough to consider the case $x := x_r$.

For (i) we have to verify that for each $L(\lambda) \in \operatorname{Irr}(\mathcal{F}_+(\mathfrak{g}))$, and $L_{\mathfrak{g}_x}(\nu) \in \operatorname{Irr}\big(\mathcal{F}(\mathfrak{g}_x)\big)$ the graded multiplicity

$$(d_0|d_1) := [\operatorname{DS}_x(L(\lambda)) : L_{\mathfrak{g}_x}(\nu)]$$

satisfies

$$\begin{cases} d_1 = 0 & \text{if} \quad \operatorname{dex}(\nu) = 1, \\ d_0 = 0 & \text{if} \quad \operatorname{dex}(\nu) = -1. \end{cases} \tag{19}$$

Recall that $\operatorname{dex}(\lambda) = \operatorname{dex}(\operatorname{howl}(\lambda))$. Using Corollary 6.4.3 we reduce (i) to the case when λ, ν are core-free. In this case $\mathfrak{g} = \mathfrak{osp}(2n + t|2n) = \mathfrak{g}_n$ (for some n) and $\mathfrak{g}_x = \mathfrak{g}_{n-r}$.

We proceed by induction on r. Note that (ii) for $r = 0$ is tautological.

Consider the case $r = 1$ for $t = 0, 1$. For a core-free diagram f denote by $||f||$ the sum of the coordinates of the symbols $\times$ in f. One has $\operatorname{dex}(\lambda(f)) = (-1)^{||f||}$, so (19) follows from Theorem 8.2. The case $r = 1$ for $t = 2$ follows from $t = 1$ and Lemma 8.4.1. This establishes (i) for $r = 1$.

Now fix any t and take $r \geq 2$. By induction, $\operatorname{DS}_{r-1}(L(\lambda)) \in \mathcal{F}_+(\mathfrak{g}_{n-r+1})$. Using (i) for $r = 1$ we get

$$N := \operatorname{DS}_1(\operatorname{DS}_{r-1}(L(\lambda)) \in \mathcal{F}_+(\mathfrak{g}_x).$$

By [16], $\operatorname{sch} N = \operatorname{sch} \operatorname{DS}_r(L(\lambda))$; by [8], Lem. 2.4.1, $\dim \operatorname{DS}_r(L(\lambda)) \leq \dim N$. Using Lemma 9.2 we obtain $N \cong \operatorname{DS}_r(L(\lambda))$ as required. $\square$

9.4. **Corollary**. *Take $\tau : \Lambda^{(2)}_{m+1|n} \to \Lambda^{(1)}_{m|n}$ as in 3.7.7. One has*

$$[\mathrm{DS}_r(L(\lambda)) : L(\mathfrak{g}_x(\nu))] = [\mathrm{DS}_r(L(\tau(\lambda))) : L(\mathfrak{g}_x(\tau(\nu)))].$$

Proof. The case $r = 1$ was treated in Lemma 8.4.1. The general case follows from Theorem 9.3 (ii). □

9.4.1. We say that a module M is *pure* if for any subquotient L of M, $\Pi(L)$ is not a subquotient of M. Theorem 9.3 implies immediately the following assertion.

9.4.2. **Corollary**. *$DS_x(L(\lambda))$ is pure for any x and any λ.*

9.4.3. **Corollary**. *For irreducible $L(\lambda)$*

$$\mathrm{DS}_1(L(\lambda)) \cong \bigoplus_i m_i \Pi^{n_i} L(\lambda_i)$$

where the arc diagram of $L(\lambda_i)$ is obtained by removing the i-th maximal arc and the associated $\times$ from the arc diagram of λ. The multiplicity m_i is 1 or 2 according to the rules of Theorem 8.2 and $n_i = 1 \bmod 2$ if and only if the parities of λ and λ_i differ.

Since $\mathrm{DS}_{r+1}(L) \cong \mathrm{DS}_1(\mathrm{DS}_r(L))$ we can calculate any $\mathrm{DS}_{r+1}(L)$ by repeated application of DS_1.

9.4.4. Remark

In the $\mathfrak{gl}(m|n)$-case $\mathrm{DS}(L(\lambda))$ is even multiplicity free [14].

9.5. The OSp-case

Using Corollary 9.4.3 it is easy to describe the effect of DS_1 on irreducible $OSp(M|N)$-modules $L_{OSp}(\lambda)$, see below.

Let $\tilde{\mathcal{F}}'(M|N)$ denote the category of algebraic representations of $OSp(M|N)$, then we have a commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{F}}'(M|N) & \xrightarrow{Res} & \tilde{\mathcal{F}}(M|N) \\ \downarrow{\scriptstyle DS_1} & & \downarrow{\scriptstyle DS_1} \\ \tilde{\mathcal{F}}'(M-2|N-2) & \xrightarrow{Res} & \tilde{\mathcal{F}}(M-2|N-2). \end{array}$$

Formula (16) holds for $OSp(2m+1|2n)$-modules if the signs of $L_{OSp}(\lambda, \pm)$ and $L_{OSp}(\nu, \pm)$ are equal; otherwise the multiplicity is zero.

Consider $OSp(2m|2n)$-case. Combining Remark 3.1.8 and Corollary 9.4.3 we conclude that $\mathrm{DS}_x(L(\lambda))$ has a structure of $OSp(2m|2n)$-module. Let us show that the multiplicity

$d_{OSp}(\lambda;\nu) := [\mathrm{DS}_1(L_{OSp}(\lambda)) : L_{OSp}(\nu)]$ is given by the formula (17) in the case $\lambda^\sigma = \lambda$, $t = 0$ and by the formula (16) otherwise.

Indeed, by Remark 3.1.8 we get

$$d_{OSp}(\lambda;\nu) = [\mathrm{DS}_1(L_{OSp}(\lambda)) : L(\nu)] = \begin{cases} [\mathrm{DS}_1(L(\lambda)) : L(\nu)] & \text{if } \lambda^\sigma = \lambda \\ 2[\mathrm{DS}_1(L(\lambda)) : L(\nu)] & \text{otherwise.} \end{cases}$$

Thus $d_{OSp}(\lambda;\nu)$ is given by the formula (17) (resp., (16)) for the case when $\lambda^\sigma = \lambda$ and $t = 0$ (resp., $t = 2$). For the remaining case $t = 0$ and $\lambda^\sigma \neq \lambda$ we get

$$d_{OSp}(\lambda;\nu) = \begin{cases} (2|0) & \text{if } e \text{ is even} \\ (0|2) & \text{if } e \text{ is odd,} \end{cases} \tag{20}$$

where e as in Theorem 8.2. Notice that $e \neq 0$, since the condition $\lambda^\sigma \neq \lambda$ implies that the zero position of the diagram of λ is empty. Hence $d_{OSp}(\lambda;\nu)$ is given by the formula (16).

9.6. Question

For a simple module L in $\tilde{\mathcal{F}}(\mathfrak{osp}(2m|2n))$, Theorem 9.3 implies that $\mathrm{DS}_x(L)$ is an $OSp(2m|2n)$-module for any x. Is this still true for an arbitrary module in $\tilde{\mathcal{F}}(2m|2n)$?

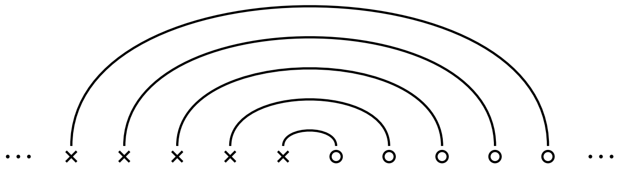
A related question can be asked about finite-dimensional modules over $\mathfrak{g} = D(2|1,a), F(4)$ (for these cases $\mathrm{DS}_x(\mathfrak{g})$ admits an involution σ and $\mathrm{DS}_x(L)$ is σ-invariant for each simple module L in $\tilde{\mathcal{F}}(\mathfrak{g})$, see [9]).

9.7. Superdimensions

Similarly to [14][19] in the $\mathfrak{gl}(m|n)$-case this allows us now to compute the superdimension of any irreducible $L(\lambda)$. Let λ be maximal atypical and x of rank equal to the atypicality. Then $\mathfrak{g}_x$ is either an orthogonal or symplectic Lie algebra or $\mathfrak{osp}(1|2r)$ for some r. In each case the superdimension of an irreducible module is known. Since DS is a symmetric monoidal functor it preserves the superdimension. So $sdim(L(\lambda)) = sdim(DS(L(\lambda)))$. So in order to compute $\mathrm{sdim}(L(\lambda))$ it suffices to compute the multiplicity $m(\lambda)$ of the isotypic representation $DS(L(\lambda))$ of $\mathfrak{g}_x$, but this multiplicity is computed exactly by Theorem 8.2 since $DS_r(L(\lambda)) = DS_1(\ldots(DS_1(L(\lambda))))$. By [10, Corollary 7.29] the number $m(\lambda)$ is also equal to the number of increasing paths in a certain graph from the Kostant weights to λ.

9.7.1. Example

Consider a core-free weight λ with the symbols $\times$ occupying adjacent positions with no $\times$ at the zero position. Then the arc diagram is completely nested:



and in each step the maximal arc is removed. For $t = 1, t = 2$ the corresponding irreducible summand occurs with multiplicity 2; for $t = 0$ the same happens with the corresponding $OSp(2m|2m)$-modules (see (20)). Since $\operatorname{sdim} L_{Osp(2m|2m)}(\lambda) = 2 \operatorname{sdim} L_{\mathfrak{osp}(2m|2n)}(\lambda)$ we obtain

$$\operatorname{sdim} L_{\mathfrak{g}}(\lambda) = \begin{cases} 1 & \text{for } \mathfrak{g} = \mathfrak{gl}(m|m) \\ 2^{m-1} & \text{for } \mathfrak{g} = \mathfrak{osp}(2m|2m) \\ 2^m & \text{for } \mathfrak{g} = \mathfrak{osp}(2m+1|2m), \mathfrak{osp}(2m+2|2m). \end{cases}$$

9.7.2. Example

Consider a core-free weight λ with the symbols $\times$ which do not occupy adjacent positions and the zero position. In this case the arc diagram consists of m maximal separated arcs:

Using results of [14] for the $\mathfrak{gl}$-case we get

$$\operatorname{sdim} L_{\mathfrak{g}}(\lambda) = \begin{cases} m! & \text{for } \mathfrak{g} = \mathfrak{gl}(m|m) \\ 2^{m-1} m! & \text{for } \mathfrak{g} = \mathfrak{osp}(2m|2m) \\ 2^m m! & \text{for } \mathfrak{g} = \mathfrak{osp}(2m+1|2m), \mathfrak{osp}(2m+2|2m). \end{cases}$$

9.7.3. Example

For an $\mathfrak{osp}(2m+1|2n)$-diagram with the empty zero position the multiplicity is $2^l |F|!/F!$ for $l = min(m|n)$ where F is the forest associated to the arc diagram of λ (see [14]). For $t = 0$ the multiplicity $m(\lambda)$ is given by a forest factorial exactly as for $\mathfrak{gl}(n|n)$, but one has to take into account that in this case a removal of one arc may produce two arc diagrams, which differ by the sign.

Appendix A. Functors Res and $\overline{Ind}$

Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra, $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra and $h \in \mathfrak{h}$ be an element with the following properties:

(H1) $\mathfrak{h}$ is the centralizer of $\mathfrak{h}_{\overline{0}}$ in $\mathfrak{g}$;

(H2) $\mathfrak{h}_{\overline{0}}$ acts diagonally on $\mathfrak{g}$;

(H3) all eigenvalues of $\operatorname{ad} h$ are real and $\mathfrak{g}^{\operatorname{ad} h} = \mathfrak{h}$.

In this case we have a usual triangular decomposition

$$\begin{array}{l}\mathfrak{g}=\mathfrak{h}\oplus(\oplus_{\alpha\in\Delta(\mathfrak{g})}\mathfrak{g}_\alpha)\quad\text{where}\\ \Delta(\mathfrak{g})\subset\mathfrak{h}_{\overline{0}}^*,\ \ \Delta(\mathfrak{g})=\Delta^+(\mathfrak{g})\coprod\Delta^-(\mathfrak{g})\\ \mathfrak{g}_\alpha:=\{g\in\mathfrak{g}|\ [h',g]=\alpha(h')g\ \text{ for all } h'\in\mathfrak{h}_{\overline{0}}\}\\ \Delta^+(\mathfrak{g}):=\{\alpha\in\Delta(\mathfrak{g})|\ \alpha(h)>0\},\quad \Delta^-(\mathfrak{g}):=\{\alpha\in\Delta(\mathfrak{g})|\ \alpha(h)<0\}.\end{array}$$

We set $\mathfrak{n}^\pm:=\oplus_{\alpha\in\Delta^\pm}\mathfrak{g}_\alpha,\quad \mathfrak{b}:=\mathfrak{h}\oplus\mathfrak{n}^+$. We consider the partial order on $\mathfrak{h}_{\overline{0}}^*$ given by

$$\lambda>\nu\quad\text{if } \nu-\lambda\in\mathbb{N}\Delta^-.$$

A.1. We fix $z\in\mathfrak{h}_{\overline{0}}^*$ satisfying

$$\alpha(z)\in\mathbb{R}_{\geq 0}\ \text{ for all } \alpha\in\Delta^+\ \text{ and } \alpha(z)\in\mathbb{R}_{\leq 0}\ \text{ for all } \alpha\in\Delta^-, \tag{21}$$

and introduce

$$\mathfrak{t}:=\mathfrak{g}^z,\quad \mathfrak{m}:=\bigoplus_{\alpha\in\Delta:\ \alpha(z)>0}\mathfrak{g}_\alpha,\quad \mathfrak{p}=\mathfrak{g}^z+\mathfrak{b}=\mathfrak{t}\rtimes\mathfrak{m}.$$

The triples $(\mathfrak{p}(z),\mathfrak{h},h)$ and $(\mathfrak{t}(z),\mathfrak{h},h)$ satisfy (H1)–(H3) and

$$\begin{array}{l}\Delta^+(\mathfrak{p})=\Delta^+(\mathfrak{g}),\qquad \Delta^+(\mathfrak{t})=\{\alpha\in\Delta^+(\mathfrak{g})|\ \alpha(z)=0\}\\ \Delta^-(\mathfrak{p})=\Delta^-(\mathfrak{t})=\{\alpha\in\Delta^-(\mathfrak{g})|\ \alpha(z)=0\}\end{array}$$

A.2. Functors Res_a *and* $\overline{\mathrm{Ind}}$

We denote by $\mathcal{O}$ the full category of finitely generated modules with a diagonal action of $\mathfrak{h}_{\overline{0}}$ and locally nilpotent action of $\mathfrak{n}$. It is easy to see that, up to a parity change, the simple $\mathfrak{h}$-modules are parametrized by $\lambda\in\mathfrak{h}_{\overline{0}}^*$; we denote by C_λ a simple $\mathfrak{h}$-module, where $\mathfrak{h}_{\overline{0}}$ acts by λ. We view $C(\lambda)$ as a $\mathfrak{b}$-module with the zero action of $\mathfrak{n}$ and set

$$M(\lambda):=\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}C(\lambda);$$

this module has a unique simple quotients which we denote by $L(\lambda)$. (The module $M(\lambda)$ is a Verma module if $\mathfrak{h}_{\overline{1}}^*=0$).

For each $a\in\mathbb{C}$ we define a functor $\mathrm{Res}_a:\mathcal{O}(\mathfrak{g})\to\mathcal{O}(\mathfrak{t})$ by

$$\mathrm{Res}_a(N):=\{v\in N|\ \ zv=av\}\ \text{ for } N\in\mathcal{O}(\mathfrak{g}).$$

We assume that

each module in $\mathcal{O}$ admits a unique maximal finite-dimensional quotient.

This holds, for example, if $\mathfrak{g}_{\overline{0}}$ is reductive, since in this case all modules in $\mathcal{O}$ have finite lengths.

View $V \in \mathcal{O}(\mathfrak{t})$ as a $\mathfrak{p}$-module with the trivial action of $\mathfrak{m}$ and consider the induced $\mathfrak{g}$-module

$$\mathrm{Ind}(V) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} V.$$

We denote by $\overline{\mathrm{Ind}}(V)$ the maximal finite-dimensional quotient of $\mathrm{Ind}(V)$.

For a semisimple $\mathfrak{h}_{\overline{0}}$-module N we denote by N_ν the weight space of the weight ν and by $\Omega(N)$ the set of weights of N.

A.3. Assumption

We fix a block $\mathcal{B}$ in the category $\mathcal{O}$ and set

$$P^+(\mathcal{B})_0 := \{\lambda \in \mathfrak{h}_{\overline{0}}^*| \ \dim L(\lambda) < \infty, \ \ L(\lambda) \in \mathcal{B} \ \text{ or } \ \Pi(L(\lambda)) \in \mathcal{B}\}.$$

We fix $\mathfrak{h}'' \subset \mathfrak{h}_{\overline{0}}$ such that

$$z \in \mathfrak{h}'', \quad [\mathfrak{h}'', \mathfrak{t}] = 0.$$

Assume that $\mu \in (\mathfrak{h}'')^*$ satisfies

$$\forall \lambda, \nu \in P^+(\mathcal{B}) \ \ \lambda \geq \nu, \ \ \lambda|_{\mathfrak{h}''} = \mu \implies \ \nu|_{\mathfrak{h}''} = \mu. \tag{22}$$

We set

$$A := \{\lambda \in P^+(\mathcal{B})| \ \lambda|_{\mathfrak{h}''} = \mu\}.$$

By above, if $\lambda \in A$ and $\nu \in P^+(\mathcal{B})$ are such that $\lambda \geq \nu$, then $\nu \in A$.

Let $\mathcal{F}_{\mathfrak{g}}(A)$ (resp., $\mathcal{F}_{\mathfrak{t}}(A)$) be the Serre subcategory of $\mathcal{O}(\mathfrak{g})$) (resp., of $\mathcal{O}(\mathfrak{t})$) generated by the simple modules $\{L(\lambda), \Pi(L(\lambda))\}_{\lambda \in A}$ (resp., by $\{L_{\mathfrak{t}}(\lambda), \Pi(L_{\mathfrak{t}}(\lambda))\}_{\lambda \in A}$).

A.4. **Proposition**. *Set $a := \mu(z)$ and* $\mathrm{Res} := \mathrm{Res}_a$.

(i) For $N \in \mathcal{F}_{\mathfrak{g}}(A)$ one has $\mathrm{Res}(N) = N^{\mathfrak{m}}$. *For $\lambda \in A$ one has* $\mathrm{Res}(L(\lambda)) = L_{\mathfrak{t}}(\lambda)$.

(ii) The restrictions of Res *and* $\overline{\mathrm{Ind}}$ *give an equivalence of categories $\mathcal{F}_{\mathfrak{g}}(A)$ and $\mathcal{F}_{\mathfrak{t}}(A)$.*

Proof. For each $\mathfrak{h}$-module M we denote by $\mathrm{Spec}(M)$ the set of z-eigenvalues. Recall that

$$\mathrm{Spec}(\mathfrak{m}) \subset \mathbb{R}_{>0}, \quad \mathrm{Spec}(\mathfrak{n}^-) \subset \mathbb{R}_{\leq 0}.$$

Identifying V with $1 \otimes V \subset \mathrm{Ind}(V)$ we obtain

$$\mathrm{Res}(\mathrm{Ind}(V)) = V \ \text{ for } V \in \mathcal{O}(\mathfrak{t}). \tag{23}$$

Take $N \in \mathcal{F}_{\mathfrak{g}}(A)$. For $\lambda \in A$ one has $\mathrm{Spec}(L(\lambda)) \subset (\mu(z) - \mathbb{R}_{\geq 0}) =: U$; this gives $\mathrm{Spec}(N) \subset U$ and implies

$$\mathrm{Res}(N) \subset N^{\mathfrak{m}}.$$

Set $\mathfrak{n}' := \mathfrak{n} \cap \mathfrak{t}$. Let $v \in N^{\mathfrak{m}}$ be a non-zero vector of weight ν. The subspace $\mathcal{U}(\mathfrak{n})v$ contains a singular weight vector v'; let λ' be the weight of v'. Since $N \in \mathcal{F}_{\mathfrak{g}}(A)$ we have $\lambda' \in A$, that is $\lambda'(z) = \mu(z)$. Since $\mathfrak{m}v = 0$ we have $\mathcal{U}(\mathfrak{n})v = \mathcal{U}(\mathfrak{n}')v$, so $v' \in \mathcal{U}(\mathfrak{n}')v$, that is $\nu(z) = \lambda'(z)$. Therefore $v \in \mathrm{Res}(N)$. Hence $\mathrm{Res}(N) = N^{\mathfrak{m}}$.

Take $\lambda \in A$ and denote by v_λ a highest weight vector in $L(\lambda)$. By above, for each non-zero vector v the space $\mathcal{U}(\mathfrak{n}')v$ contains v_λ. Hence $L(\lambda)^{\mathfrak{m}}$ is simple, so $L(\lambda)^{\mathfrak{m}} = L_{\mathfrak{t}}(\lambda)$. This establishes (i).

Clearly, Res is exact. By (i), Res maps simple modules in $\mathcal{F}_{\mathfrak{g}}(A)$ to simple modules in $\mathcal{F}_{\mathfrak{t}}(A)$. Therefore the restriction of Res gives an exact functor

$$\mathrm{Res} : \mathcal{F}_{\mathfrak{g}}(A) \to \mathcal{F}_{\mathfrak{t}}(A).$$

Take a module $V \in \mathcal{F}_{\mathfrak{t}}(A)$. Note that for $\nu \in A$ the module $\mathrm{Ind}\, L_{\mathfrak{t}}(\nu)$ lies in $\mathcal{B}$; by (22) each simple finite-dimensional subquotient of $\mathrm{Ind}\, L_{\mathfrak{t}}(\nu)$ is of the form $L(\nu')$, where $\nu' \in A$. Since Ind is exact, each simple subquotient of $\mathrm{Ind}(V)$ is $L(\nu')$ or $\Pi(L(\nu'))$ for some $\nu' \in A$, so $\overline{\mathrm{Ind}}(V) \in \mathcal{F}_{\mathfrak{g}}(A)$. Moreover, using the exactness of Res and (23) we get

$$[\mathrm{Ind}(V) : L(\nu)] = [V : L_{\mathfrak{t}}(\nu)],$$

that is

$$[\overline{\mathrm{Ind}}(V) : L(\nu)] \leq [V : L_{\mathfrak{t}}(\nu)]. \tag{24}$$

The module $\mathrm{Ind}(L_{\mathfrak{t}}(\lambda))$ is a quotient of $M(\lambda)$, so $L(\lambda)$ is a quotient of $\mathrm{Ind}(L_{\mathfrak{t}}(\lambda))$. Using (24) we get

$$\overline{\mathrm{Ind}}(L_{\mathfrak{t}}(\lambda)) = L(\lambda).$$

For each $V \in \mathcal{F}_{\mathfrak{t}}(A)$ and $N \in \mathcal{F}_{\mathfrak{g}}(A)$ we have

$$\mathrm{Hom}_{\mathfrak{g}}(\overline{\mathrm{Ind}}(V), N) = \mathrm{Hom}_{\mathfrak{g}}(\mathrm{Ind}(V), N) = \mathrm{Hom}_{\mathfrak{p}}(V, N) = \mathrm{Hom}_{\mathfrak{t}}(V, N^{\mathfrak{m}}).$$

Using (i) we conclude that $\overline{\mathrm{Ind}} : \mathcal{F}_{\mathfrak{t}}(A) \to \mathcal{F}_{\mathfrak{g}}(A)$ is a left adjoint to $\mathrm{Res} : \mathcal{F}_{\mathfrak{g}}(A) \to \mathcal{F}_{\mathfrak{t}}(A)$; by above, these functors map simple modules to simple modules and Res is exact.

Take any $N \in \mathcal{F}_{\mathfrak{g}}(A)$ and set $V := \mathrm{Res}(N)$. Let $\phi \in \mathrm{Hom}_{\mathfrak{g}}(\overline{\mathrm{Ind}}(V), N)$ be the preimage of the identity map

$$V \xrightarrow{\sim} \mathrm{Res}(N).$$

The image of ϕ is the submodule of N generated by $\mathrm{Res}(N)$; since Res is exact and $\mathrm{Res}(M) \neq 0$ for each $M \in \mathcal{F}(A)$, ϕ is surjective. Moreover, for each $\nu \in A$ one has

$$[N : L(\nu)] = [V : L_{\mathfrak{t}}(\nu)].$$

Combining with (24) we conclude that ϕ is bijective and for each $\nu \in A$ one has

$$[\overline{\mathrm{Ind}}(V) : L(\nu)] = [V : L_{\mathfrak{t}}\nu)]$$

which gives

$$[\mathrm{Res}(\overline{Ind}(V)) : L_{\mathfrak{t}}(\nu)] = [V : L_{\mathfrak{t}}(\nu)]. \tag{25}$$

Take $V \in \mathcal{F}_{\mathfrak{t}}(A)$. Identifying V and $1 \otimes V \subset \mathrm{Ind}\, V$ we have $V = \mathrm{Res}(\mathrm{Ind}\, V)$. Since Res is an exact functor on $\mathcal{O}$ this gives the natural surjective map $V \to \mathrm{Res}(\overline{\mathrm{Ind}}(V))$. By (25) this map is bijective. Hence Res and $\overline{\mathrm{Ind}}$ provide an equivalence of categories $\mathcal{F}_{\mathfrak{g}}(A)$ and $\mathcal{F}_{\mathfrak{t}}(A)$. $\square$

A.5. Remark

Consider the case $\mathfrak{h} = \mathfrak{h}_{\overline{0}}$ and $\mathfrak{t} = \mathfrak{l} \times \mathfrak{h}''$. Set $\mathfrak{h}' := \mathfrak{l} \cap \mathfrak{h}$ and

$$A' := \{\lambda|_{\mathfrak{h}'} |\ \lambda \in A\}.$$

Since $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$ the map $\lambda \mapsto \lambda|_{\mathfrak{h}'}$ gives a bijection between A and A'. Using Proposition A.4 we obtain the equivalence of the category $\mathcal{F}_{\mathfrak{g}}(A)$ and the Serre category $\mathcal{F}_{\mathfrak{l}}(A')$ which is generated by the simple $\mathfrak{l}$-modules $\{L_{\mathfrak{l}}(\lambda'), \Pi(L_{\mathfrak{l}}(\lambda'))\}_{\lambda' \in A'}$.

References

[1] J. Brundan, C. Stroppel, Highest weight categories arising from Khovanov's diagram algebra. IV: The general linear supergroup, J. Eur. Math. Soc. 14 (2012) 2.

[2] J. Comes, T. Heidersdorf, Thick ideals in Deligne's category $\underline{\mathrm{Rep}}(O_\delta)$, J. Algebra 480 (2017).

[3] M. Duflo, V. Serganova, On associated variety for Lie superalgebras, arXiv:math/0507198 [math.RT].

[4] M. Ehrig, C. Stroppel, On the category of finite-dimensional representations of $OSP(r|2n)$, in: H. Krause, et al. (Eds.), Representation Theory – Current Trends and Perspectives, in: EMS Ser. Congr. Rep., 2017, pp. 109–170.

[5] M. Ehrig, C. Stroppel, On the category of finite-dimensional representations of $OSP(r|2n)$, Part II, available at, http://www.math.uni-bonn.de/ag/stroppel/OSPII.pdf.

[6] M. Ehrig, C. Stroppel, Nazarov-Wenzl algebras, coideal subalgebras and categorified skew Howe duality, Adv. Math. 331 (2018).

[7] I. Entova-Aizenbud, V. Serganova, Duflo-Serganova functor and superdimension formula for the periplectic Lie superalgebra, arXiv:1910.02294, 2019, to appear in Algebra Number Theory.

[8] M. Gorelik, Depths and cores in the light of DS-functors, arXiv:2010.05721, 2020.

[9] M. Gorelik, Bipartite extension graphs and the DS functor, arXiv:2010.12817, 2020.

[10] M. Gorelik, T. Heidersdorf, Gruson-Serganova character formula and the Duflo-Serganova cohomology functor, arXiv:2104.12634, 2021.

[11] C. Gruson, V. Serganova, Cohomology of generalized supergrassmanians and character formulae for basic classical Lie superalgebras, Proc. Lond. Math. Soc. (3) 101 (2010) 852–892.
[12] C. Gruson, V. Serganova, Bernstein-Gelfand-Gelfand reciprocity and indecomposable projective modules for classical algebraic supergroups, Mosc. Math. J. 13 (2) (2013) 281–313.
[13] T. Heidersdorf, On supergroups and their semisimplified representation categories, Algebr. Represent. Theory 22 (2019) 937–959.
[14] T. Heidersdorf, R. Weissauer, Cohomological tensor functors on representations of the general linear supergroup, Mem. Am. Math. Soc. 270 (2021) 1320, arXiv:1406.0321.
[15] T. Heidersdorf, R. Weissauer, On classical tensor categories attached to the irreducible representations of the general linear supergroups $GL(n|n)$, arXiv:1805.00384, 2018.
[16] C. Hoyt, S. Reif, Grothendieck rings for Lie superalgebras and the Duflo-Serganova functor, Algebra Number Theory 12 (9) (2018) 2167–2184.
[17] V. Serganova, On a superdimension of an irreducible representation of a basic classical Lie superalgebras, in: Supersymmetry in Mathematics and Physics, in: Lecture Notes in Math., vol. 2027, Springer, Heidelberg, 2011, pp. 253–273.
[18] V. Serganova, Finite dimensional representations of algebraic supergroups, in: Proceedings of the International Congress of Mathematicians ICM 2014, Seoul, Korea, August 13–21, 2014, vol. I: Plenary Lectures and Ceremonies, 2014.
[19] R. Weissauer, Model structures, categorical quotients and representations of super commutative Hopf algebras II, The case $GL(m|n)$, arXiv:1010.3217.